Advanced Control Systems
Detection, Estimation, and Filtering

Graduate Course on the
MEng PhD Program
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Chapter 2
Minimum Variance Unbiased Estimation

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Syllabus:

Classical Estimation Theory

Chap. 1 - ***Motivation for Estimation in Stochastic Signal Processing*** [1/2 week] Motivating examples of signals and systems in detection and estimation problems;

Chap. 2 - ***Minimum Variance Unbiased Estimation*** [1/2 week]

Unbiased estimators; Minimum Variance Criterion; Extension to vector parameters; Efficiency of estimators;

Chap. 3 - ***Cramer-Rao Lower Bound*** [1 week]
Estimator accuracy; Cramer-Rao lower bound (CRLB); CRLB for signals in white Gaussian noise; Examples;

continues…

Unbiased estimators:

The search for good estimators for unknown deterministic parameters begins

Example (revisited):

Unbiased estimator for DC level in white Gaussian noise. Signal model is

$$x[n] = A + w[n] \qquad n = 0,1,\ldots,N-1$$

A reasonable estimator is $\hat{A} = k\frac{1}{N}\sum_{n=0}^{N-1} x[n]$

Due to the linearity properties of the expectation operator *E[.]*:

$$E\left[\hat{A}\right] = E\left[k\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right] = k\frac{1}{N}\sum_{n=0}^{N-1} E\left[x[n]\right] = k\frac{1}{N}NA = kA$$

Unbiased estimator iff k=1!

Unbiasness:

Let the vector of deterministic unknown parameters, with p components, be described as

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{bmatrix} = \begin{bmatrix} \theta_1 & \theta_2 & \dots & \theta_p \end{bmatrix}^T$$

An estimator must have the same dimensions, i.e.

$$\hat{\theta} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \vdots \\ \hat{\theta}_p \end{bmatrix} = \begin{bmatrix} \hat{\theta}_1 & \hat{\theta}_2 & \dots & \hat{\theta}_p \end{bmatrix}^T$$

The estimator is **unbiased** iff

$$E\left[\hat{\theta}\right] = \theta$$

Minimum variance criterion:

In searching for estimators some optimality criterion must be adopted.

A natural one is the **mean square error (MSE),** defined as

$$mse\left(\hat{\theta}\right)=E\left[\left(\hat{\theta}-\theta\right)^2\right]$$

Unfortunately the choice of the criterion leads to unrealizable estimators, i.e. not only function of the data

$$mse\left(\hat{\theta}\right)=E\left\{\left[\left(\hat{\theta}-E\left[\hat{\theta}\right]\right)+\left(E\left[\hat{\theta}\right]-\theta\right)\right]^2\right\}=\mathrm{var}\left(\hat{\theta}\right)+\left[\left(E\left[\hat{\theta}\right]-\theta\right)\right]^2=$$

$$=\mathrm{var}\left(\hat{\theta}\right)+b^2\left(\theta\right),\qquad where\quad b^2\left(\theta\right)=E\left[\hat{\theta}\right]-\theta.$$

Fortunately, after differentiation an estimator depending on θ will result.

Minimum variance criterion:

Example:

Find the value of k such that a realizable mse estimator results

$$x[n] = A + w[n] \qquad n = 0,1,\dots,N-1, \qquad \hat{A} = k\frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

From the previous page

$$\mathrm{var}\left(\hat{\mathrm{A}}\right) = \frac{k^2\sigma^2}{N}, \qquad and \qquad b^2(A) = E\left[\hat{\mathrm{A}}\right] - A = (kA - A)^2$$

$$mse\left(\hat{\mathrm{A}}\right) = \mathrm{var}\left(\hat{\mathrm{A}}\right) + b^2(A) = \frac{k^2\sigma^2}{N} + (k-1)^2 A^2$$

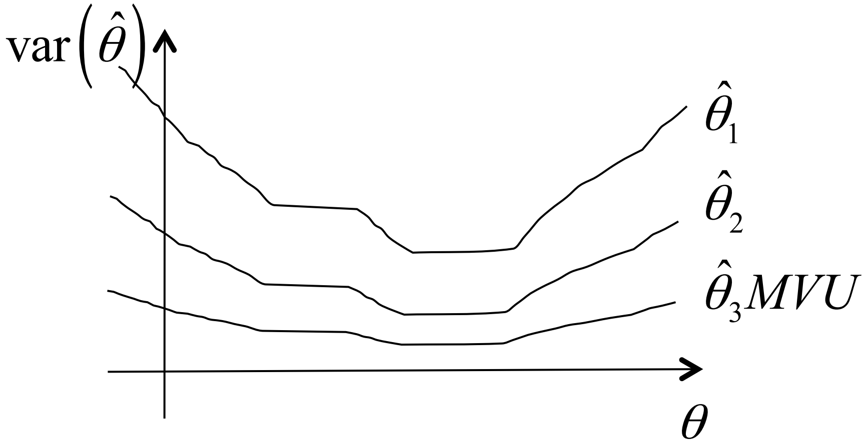
Lets find the minimum

$$\frac{d}{dk} mse\left(\hat{\mathrm{A}}\right) = \frac{2k\sigma^2}{N} + 2(k-1)A^2 = 0,$$ results in $$k_{opt} = \frac{A^2}{A^2 + \sigma^2 / N}$$

Unfortunately depends on the unknown parameter A.

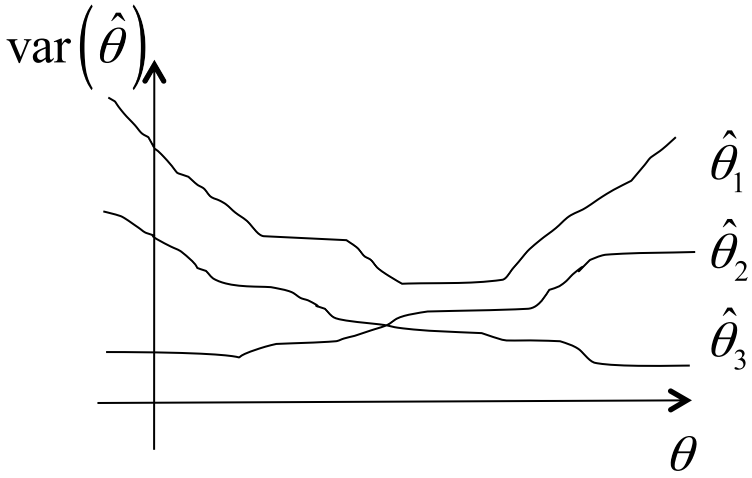
Minimum variance unbiased estimator:

The Minimum Variance Unbiased (MVU) Estimator **must have** smallest variance for all values of θ.



In general, the MVU estimator does not always exist.

There is no “turn-the-crank ” method.



Future approaches:

Chp. 3 – Cramer-Rao lower bound

Chp. 5 – Sufficient statistics

Chp. 6 – Restrict to linear estimators: BLUE

Bibliography:

Further reading

• Harry L. Van Trees, ***Detection, Estimation, and Modulation Theory, Parts I to IV,*** John Wiley, 2001.

• J. Bibby, H. Toutenburg, ***Prediction and Improved Estimation in Linear Models,*** John Wiley, 1977.

• C.Rao, ***Linear Statistical Inference and Its Applications,*** John Wiley, 1973.

• P. Stoica, R. Moses, *“On Biased Estimators and the Unbiased Cramer-Rao Lower Bound,” Signal Process,* vol.21, pp. 349-350, 1990.