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STOCHASTIC ESTIMATION

The Steady State Kalman Filter:
Discrete-Time Case.

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MOTIVATION

We seek special properties of discrete-time Kalman filters when

- the state-dynamics are linear and time invariant
- the measurement equation is linear and time invariant
- the noise statistics are stationary

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SUMMARY OF RESULTS

- 1) As $t \rightarrow \infty$, the error covariance matrix $\underline{\Sigma}(t) \rightarrow \underline{\Sigma} = \text{constant}$
- 2) The filter gain matrix $\underline{H}(t) \rightarrow \underline{H} = \text{constant}$
- 3) The Kalman filter is linear and time-invariant

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PROBLEM FORMULATION

- State Dynamics

$$\underline{x}(t+1) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) + \underline{L}\underline{\xi}(t) \quad (1)$$

- Measurement Equation

$$\underline{z}(t+1) = \underline{C}\underline{x}(t+1) + \underline{e}(t+1) \quad (2)$$

- \underline{A} , \underline{B} , \underline{L} , \underline{C} constant known matrices

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PROBABILISTIC ASSUMPTIONS

- Initial Time: $t=0$
- Initial State: $\underline{x}(0)$ gaussian random vector

$$E \{ \underline{x}(0) \} = \underline{\bar{x}}(0) \quad (3)$$

$$\text{cov} [\underline{x}(0); \underline{x}(0)] = \underline{\Sigma}(0) = \underline{\Sigma}'(0) \geq \underline{0} \quad (4)$$

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Plant Noise $\underline{\xi}(t)$ is assumed to be stationary, white, gaussian noise

$$E \{ \underline{\xi}(t) \} = \underline{0} \quad \text{all } t=0, 1, 2, \dots \quad (5)$$

$$\text{cov} [\underline{\xi}(t); \underline{\xi}(\tau)] = \underline{\Xi} \delta_{t\tau} \quad (6)$$

$$\underline{\Xi} = \underline{\Xi}' > \underline{0} \quad (\underline{\Xi} = \text{constant matrix}) \quad (7)$$

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Measurement Noise $\underline{e}(t)$, is assumed to be stationary, white, gaussian noise

$$E \{ \underline{e}(t) \} = \underline{0} \quad \text{all } t=1, 2, \dots \quad (8)$$

$$\text{cov} [\underline{e}(t); \underline{e}(\tau)] = \underline{\Theta} \delta_{t\tau} \quad (9)$$

$$\underline{\Theta} = \underline{\Theta}' > \underline{0} \quad (\underline{\Theta} = \text{constant matrix}) \quad (10)$$

Additional assumption:

$$\underline{x}(0), \underline{\xi}(t), \underline{e}(\tau)$$

mutually independent for all t, τ

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SPECIAL FORM OF KALMAN FILTER

We can obtain using previous results the discrete-time Kalman filter for the special case

$\underline{A}, \underline{B}, \underline{C}, \underline{L}, \underline{\Xi}, \underline{\Theta}$ = constant matrices

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We shall also make additional assumptions

Controllability

$$[A, L] = \text{controllable}$$

(11) white noise $\xi(t)$ must excite all dynamic modes

i.e.

$$\text{rank} \begin{bmatrix} L & AL & A^2L & \dots & A^{n-1}L \end{bmatrix} = n \quad (12)$$

Observability

$$[A, C] = \text{observable}$$

(13) all dynamic modes must be observable in noiseless measurements.

i.e.

$$\text{rank} \begin{bmatrix} C & A'C & A^2C & \dots & A^{n-1}C \end{bmatrix} = n \quad (14)$$

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ERROR COVARIANCE MATRIX

• Initialization

$$\underline{\Sigma}(0/0) = \underline{\Sigma}(0) = \text{cov} [\underline{x}(0); \underline{x}(0)] \quad (15)$$

• Predict Cycle

$$\underline{\Sigma}(t+1/t) = \underline{A} \underline{\Sigma}(t/t) \underline{A}' + \underline{L} \underline{\Xi} \underline{L}' \quad (16)$$

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• Update Cycle

$$\begin{aligned} \underline{\Sigma}(t+1/t+1) &= \underline{\Sigma}(t+1/t) - \underline{\Sigma}(t+1/t) \underline{C}' \\ &\quad \cdot \left[\underline{C} \underline{\Sigma}(t+1/t) \underline{C}' + \underline{\Theta} \right]^{-1} \underline{C} \underline{\Sigma}(t+1/t) \end{aligned} \quad (17)$$

• Filter Gain Matrix

$$\underline{H}(t+1) = \underline{\Sigma}(t+1/t+1) \underline{C}' \underline{\Theta}^{-1} \quad (18)$$

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ASYMPTOTIC PROPERTIES

- As $t \rightarrow \infty$, the error covariance matrix $\underline{\Sigma}(t/t)$ tends to a constant matrix $\underline{\Sigma}$, i.e.

$$\lim_{t \rightarrow \infty} \underline{\Sigma}(t/t) = \underline{\Sigma} \quad (19)$$

- $\underline{\Sigma}$ is called the steady-state error covariance matrix

As more and more measurements are taken, initial transients, due to $\underline{\Sigma}(0)$, die-out.

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• Let

$$\underline{\Sigma}_p = \lim_{t \rightarrow \infty} \underline{\Sigma}(t+1/t) \quad (20)$$

From predict equation

$$\underline{\Sigma}_p = \underline{A} \underline{\Sigma}_p \underline{A}' + \underline{L} \underline{\Xi} \underline{L}' \quad (21)$$

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• From update equation

$$\underline{\Sigma} = \underline{\Sigma}_p - \underline{\Sigma}_p \underline{C}' \left[\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta} \right]^{-1} \underline{C} \underline{\Sigma}_p \quad (22)$$

• Combine predict and update eqs (21), (22)

$$\begin{aligned} \underline{0} &= -\underline{\Sigma}_p + \underline{A} \underline{\Sigma}_p \underline{A}' + \underline{L} \underline{\Xi} \underline{L}' \\ &\quad - \underline{A} \underline{\Sigma}_p \underline{C}' \left[\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta} \right]^{-1} \underline{C} \underline{\Sigma}_p \underline{A}' \end{aligned} \quad (23)$$

• Eq. (23) is called the discrete algebraic matrix Riccati equation

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PROPERTIES OF $\underline{\Sigma}$, $\underline{\Sigma}_p$

- Both are symmetric and positive definite matrices.
- $\underline{\Sigma}_p$ is the unique positive definite solution of the discrete algebraic matrix Riccati equation.

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PROPERTIES OF FILTER GAIN MATRIX

$\underline{H}(t) = KF$ gain

Recall

$$\underline{H}(t) = \underline{\Sigma}(t/t) \underline{C}' \underline{\Theta}^{-1} \quad (24)$$

Since

$$\underline{\Sigma} = \lim_{t \rightarrow \infty} \underline{\Sigma}(t/t) \quad (25)$$

Then

$$\underline{H} = \lim_{t \rightarrow \infty} \underline{H}(t) = \text{constant} \quad (26)$$

$$\underline{H} = \underline{\Sigma} \underline{C}' \underline{\Theta}^{-1}$$

(27) KF gain approaches a constant matrix

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ALTERNATE FORM FOR \underline{H}

From Eqs (27) and (22)

$$\begin{aligned} \underline{H} &= \underline{\Sigma}_p \underline{C}' \underline{\Theta}^{-1} - \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} \underline{C} \underline{\Sigma}_p \underline{C}' \underline{\Theta}^{-1} \\ &= \underline{\Sigma}_p \underline{C}' \left[\underline{I} - [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} \underline{C} \underline{\Sigma}_p \underline{C}' \right] \underline{\Theta}^{-1} \\ &= \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} [\cancel{\underline{C} \underline{\Sigma}_p \underline{C}'} + \underline{\Theta} - \cancel{\underline{C} \underline{\Sigma}_p \underline{C}'}] \underline{\Theta}^{-1} \\ &= \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} \underbrace{[\underline{\Theta} \underline{\Theta}^{-1}]}_{\underline{I}} \end{aligned} \quad (28)$$

Hence

$$\underline{H} = \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1}$$

(29) KF gain calculation using steady-state predicted covariance, $\underline{\Sigma}_p$

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THE STEADY STATE ESTIMATE

(optimal only at steady-state, under Contr. + observ. assumptions).

- Predict Equation

$$\hat{\underline{x}}(t+1/t) = \underline{A} \hat{\underline{x}}(t/t) + \underline{B} \underline{u}(t) \quad (30)$$

- Update Equation

$$\begin{aligned} \hat{\underline{x}}(t+1/t+1) &= \hat{\underline{x}}(t+1/t) \\ &+ \underline{H} [\underline{z}(t+1) - \underline{C} \hat{\underline{x}}(t+1/t)] \end{aligned} \quad (31)$$

A can be unstable!

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- Combine predict and update equations

$$\begin{aligned} \hat{\underline{x}}(t+1/t+1) = & [\underline{A} - \underline{H} \underline{C} \underline{A}] \hat{\underline{x}}(t/t) \\ & + [\underline{B} - \underline{H} \underline{C} \underline{B}] \underline{u}(t) \\ & + \underline{H} \underline{z}(t+1) \end{aligned} \quad (32)$$

- The Kalman filter system matrix $[\underline{A} - \underline{H} \underline{C} \underline{A}]$ has all eigenvalues less than unity
- Hence, steady state Kalman filter is asymptotically stable in the large

A can be unstable!

← consequence of controllability and observability assumptions

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SPECIAL CASE NO PLANT NOISE

- If

$$\underline{\xi}(t) = \underline{0} \quad (\text{set } \underline{\Xi} = \underline{0} \text{ or } \underline{L} = \underline{0})$$

then

$$\begin{aligned} \underline{\Sigma}(t+1/t+1) = & \underline{A} [\underline{\Sigma}(t/t) - \underline{\Sigma}(t/t) \underline{A}' \underline{C}'] \\ & \left(\underline{C} \underline{A} \underline{\Sigma}(t/t) \underline{A}' \underline{C}' + \underline{0} \right)^{-1} \underline{C} \underline{A} \underline{\Sigma}(t/t) \underline{A}' \end{aligned} \quad (33)$$

violates controllability assumption!

(33) MUST USE TIME-VARYING KF!

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- Under observability assumptions

$$\lim_{t \rightarrow \infty} \underline{\Sigma}(t/t) = \underline{\Sigma} = \underline{0}$$

$$\lim_{t \rightarrow \infty} \hat{\underline{x}}(t/t) = \underline{x}(t)$$

(35) with many many measurements, we remove all state uncertainty!

(36)

Hence, perfect state estimation occurs

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- Note that

$$\lim_{t \rightarrow \infty} \underline{H}(t) = \underline{0} \quad (37)$$

i. e. asymptotically Kalman filter runs "open-loop"

- It makes no sense to implement steady-state Kalman filter

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- To compensate for uncertainties in \underline{A} , \underline{B} , \underline{L} , \underline{C} one must use a non-zero value for $\underline{\Xi}$
- This tends to prevent divergence of Kalman filter.

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SUMMARY OF CALCULATIONS

• Off-Line Computations

Step 1: Determine unique positive definite solution matrix $\underline{\Sigma}_p$ of algebraic matrix equation - see eq. (23)

$$\underline{0} = -\underline{\Sigma}_p + \underline{A} \underline{\Sigma}_p \underline{A}' + \underline{L} \underline{\Xi} \underline{L}' \quad (38)$$

$$- \underline{A}' \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} \underline{C} \underline{\Sigma}_p \underline{A}$$

$$\underline{\Sigma}_p = \lim_{t \rightarrow \infty} \underline{\Sigma}(t+1/t) \quad (39)$$

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Step 2:(Optional) Find

$$\underline{\Sigma} = \underline{\Sigma}_p - \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} \underline{C} \underline{\Sigma}_p \quad (40)$$

$$\underline{\Sigma} = \lim_{t \rightarrow \infty} \underline{\Sigma}(t/t) \quad (41)$$

- $\underline{\Sigma}$ is useful to deduce RMS state estimation accuracy a priori

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Step 3: Calculate steady state filter gain matrix \underline{H}

$$\underline{H} = \underline{\Sigma}_p \underline{C}' [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}]^{-1} \quad (42)$$

or

$$\underline{H} = \underline{\Sigma} \underline{C}' \underline{\Theta}^{-1} \quad (43)$$

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• On-Line Calculations

Step 4: Construct state estimate

$$\hat{\underline{x}}(t+1/t+1) = [\underline{A} - \underline{H}\underline{C}\underline{A}] \hat{\underline{x}}(t/t) + [\underline{B} - \underline{H}\underline{C}\underline{B}] \underline{u}(t) + \underline{H}\underline{z}(t+1) \quad (44)$$

$$\hat{\underline{x}}(0/0) = E \{ \underline{x}(0) \} \quad (45)$$

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Note: Residual sequence (at steady-state !)

$$\begin{aligned} \underline{r}(t+1) &\triangleq \underline{z}(t+1) - \underline{C} \hat{\underline{x}}(t+1/t) \\ &= \underline{z}(t+1) - \underline{C} [\underline{A} \hat{\underline{x}}(t/t) + \underline{B} \underline{u}(t)] \end{aligned} \quad (46)$$

is stationary, gaussian, zero mean
white with

$$\text{cov} [\underline{r}(t); \underline{r}(\tau)] = [\underline{C} \underline{\Sigma}_p \underline{C}' + \underline{\Theta}] \delta_{t\tau} \quad (47)$$



Finite residual covariance;
uncorrelated in time!