REF. NO. KF #10

1
STOCHASTIC ESTIMATION

The Steady State Kalman Filter:
Discrete-Time Case.

2
MOTIVATION

We seek special properties of discrete-time Kalman filters when

- the state-dynamics are linear and time invariant
- the measurement equation is linear and time invariant
- the noise statistics are stationary

3
SUMMARY OF RESULTS

1) As $t \rightarrow \infty$, the error covariance matrix $\underline{\Sigma}(t) \rightarrow \underline{\Sigma}$ = constant
2) The filter gain matrix $\underline{H}(t) \rightarrow \underline{H}$ = constant
3) The Kalman filter is linear and time-invariant

4
PROBLEM FORMULATION

- State Dynamics

$$\underline{x}(t+1) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) + \underline{L}\,\underline{\xi}(t) \qquad (1)$$

- Measurement Equation

$$\underline{z}(t+1) = \underline{C}\,\underline{x}(t+1) + \underline{\theta}(t+1) \qquad (2)$$

- $\underline{A}$, $\underline{B}$, $\underline{L}$, $\underline{C}$ constant known matrices

5

PROBABILISTIC ASSUMPTIONS

- Initial Time: $t=0$
- Initial State: $\underline{x}(0)$ gaussian random vector

$$E\left\{\underline{x}(0)\right\} = \overline{\underline{x}}(0) \quad (3)$$

$$\text{cov}\left[\underline{x}(0);\underline{x}(0)\right] = \underline{\Sigma}(0) = \underline{\Sigma}'(0) \geq \underline{0} \quad (4)$$

6

Plant Noise $\underline{\xi}(t)$ is assumed to be stationary, white, gaussian noise

$$E\left\{\underline{\xi}(t)\right\} = \underline{0} \quad \text{all } t=0,1,2,\ldots \quad (5)$$

$$\text{cov}\left[\underline{\xi}(t);\underline{\xi}(\tau)\right] = \underline{\Xi}\,\delta_{t\tau} \quad (6)$$

$$\underline{\Xi} = \underline{\Xi}' > \underline{0} \quad (\underline{\Xi} = \text{constant matrix}) \quad (7)$$

7

Measurement Noise $\underline{\theta}(t)$, is assumed to be stationary, white, gaussian noise

$$E\left\{\underline{\theta}(t)\right\} = \underline{0} \quad \text{all } t=1,2,\ldots \quad (8)$$

$$\text{cov}\left[\underline{\theta}(t);\ \underline{\theta}(t)\right] = \underline{\Theta}\,\delta_{t\tau} \quad (9)$$

$$\underline{\Theta} = \underline{\Theta}' > \underline{0} \quad \left(\underline{\Theta} = \text{constant matrix}\right) \quad (10)$$

Additional assumption:

$\underline{x}(0)$, $\underline{\xi}(t)$, $\underline{\theta}(\tau)$

mutually independent for all t, τ

8

SPECIAL FORM OF KALMAN FILTER

We can obtain using previous results the discrete-time Kalman filter for the special case

$\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{L}$, $\underline{\Xi}$, $\underline{\Theta}$ = constant matrices

9

We shall also make additional assumptions

Controllability

$$[\underline{A}, \underline{L}] = \text{controllable} \tag{11}$$

white noise $\xi(t)$ must excite all dynamic modes

i. e.

$$\text{rank}\ [\underline{L} \mid \underline{A}\underline{L} \mid \underline{A}^2\underline{L} \mid \dots \mid \underline{A}^{n-1}\underline{L}] = n \tag{12}$$

Observability

$$[\underline{A}, \underline{C}] = \text{observable} \tag{13}$$

all dynamic modes must be observable in noiseless measurements.

i. e.

$$\text{rank}\ [\underline{C}' \mid \underline{A}'\underline{C}' \mid \underline{A}'^2\underline{C}' \mid \dots \mid \underline{A}'^{n-1}\underline{C}'] = n \tag{14}$$

10

ERROR COVARIANCE MATRIX

• Initialization

$$\underline{\Sigma}(0/0) = \underline{\Sigma}(0) = \text{cov}\ [\underline{x}(0); \underline{x}(0)] \tag{15}$$

• Predict Cycle

$$\boxed{\underline{\Sigma}(t+1/t) = \underline{A}\ \underline{\Sigma}(t/t)\,\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}'} \tag{16}$$

11

• Update Cycle

$$\underline{\Sigma}(t+1/t+1) = \underline{\Sigma}(t+1/t) - \underline{\Sigma}(t+1/t)\,\underline{C}' \cdot \left[\underline{C}\,\underline{\Sigma}(t+1/t)\,\underline{C}' + \underline{\Theta}\right]^{-1} \underline{C}\,\underline{\Sigma}(t+1/t) \tag{17}$$

• Filter Gain Matrix

$$\boxed{\underline{H}(t+1) = \underline{\Sigma}(t+1/t+1)\,\underline{C}'\,\underline{\Theta}^{-1}} \tag{18}$$

12

ASYMPTOTIC PROPERTIES

- As $t \to \infty$, the error covariance matrix $\underline{\Sigma}(t/t)$ tends to a constant matrix $\underline{\Sigma}$, i. e.

$$\lim_{t \to \infty} \underline{\Sigma}(t/t) = \underline{\Sigma} \tag{19}$$

- $\underline{\Sigma}$ is called the steady-state error covariance matrix

As more and more measurements are taken, initial transients, due to $\Sigma(0)$, die-out.

13

- Let

$$\underline{\Sigma}_p = \lim_{t \to \infty} \underline{\Sigma}(t+1/t) \qquad (20)$$

From predict equation

$$\boxed{\underline{\Sigma}_p = \underline{A}\,\underline{\Sigma}\,\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}'} \qquad (21)$$

14

- From update equation

$$\boxed{\underline{\Sigma} = \underline{\Sigma}_p - \underline{\Sigma}_p\underline{C}'\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}\underline{C}\,\underline{\Sigma}_p} \qquad (22)$$

- Combine predict and update eqs (21), (22)

$$\boxed{\begin{aligned}\underline{0} = &-\underline{\Sigma}_p + \underline{A}\,\underline{\Sigma}_p\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}' \\ &- \underline{A}\,\underline{\Sigma}_p\underline{C}'\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}\underline{C}\,\underline{\Sigma}_p\underline{A}'\end{aligned}} \qquad (23)$$

- Eq. (23) is called the discrete algebraic matrix Riccati equation

15

PROPERTIES OF $\underline{\Sigma}$, $\underline{\Sigma}_p$

- Both are symmetric and positive definite matrices.
- $\underline{\Sigma}_p$ is the unique positive definite solution of the discrete algebraic matrix Riccati equation.

16

PROPERTIES OF FILTER GAIN MATRIX

$\underline{H}(t) = \text{KF gain}$

Recall

$$\underline{H}(t) = \underline{\Sigma}(t/t)\underline{C}'\underline{\Theta}^{-1} \quad (24)$$

Since

$$\underline{\Sigma} = \lim_{t \to \infty} \underline{\Sigma}(t/t) \quad (25)$$

Then

$$\underline{H} = \lim_{t \to \infty} \underline{H}(t) = \text{constant} \quad (26)$$

$$\boxed{\underline{H} = \underline{\Sigma}\,\underline{C}'\,\underline{\Theta}^{-1}} \quad (27)$$

KF gain approaches a constant matrix

17

ALTERNATE FORM FOR H

From Eqs (27) and (22)

$$\underline{H} = \underline{\Sigma}_p\underline{C}'\underline{\Theta}^{-1} - \underline{\Sigma}_p\underline{C}'\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}\underline{C}\,\underline{\Sigma}_p\underline{C}'\underline{\Theta}^{-1} \quad (28)$$

$$= \underline{\Sigma}_p\underline{C}'\left[\underline{I} - \left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}\underline{C}\,\underline{\Sigma}_p\underline{C}'\right]\underline{\Theta}^{-1}$$

$$= \underline{\Sigma}_p\underline{C}'\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta} - \underline{C}\,\underline{\Sigma}_p\underline{C}'\right]\underline{\Theta}^{-1}$$

$$= \underline{\Sigma}_p\underline{C}'\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}\underbrace{\underline{\Theta}\,\underline{\Theta}^{-1}}_{\underline{I}}$$

Hence

$$\boxed{\underline{H} = \underline{\Sigma}_p\underline{C}'\left[\underline{C}\,\underline{\Sigma}_p\underline{C}' + \underline{\Theta}\right]^{-1}} \quad (29)$$

KF gain calculation using steady-state predicted covariance, $\underline{\Sigma}_p$

18

THE STEADY STATE ESTIMATE

(optimal only at steady-state, under contr. + observ. assumptions).

- Predict Equation

$$\boxed{\hat{\underline{x}}(t+1/t) = \underline{A}\hat{\underline{x}}(t/t) + \underline{B}\underline{u}(t)} \quad (30)$$

$\underline{A}$ can be unstable!

- Update Equation

$$\boxed{\begin{aligned}\hat{\underline{x}}(t+1/t+1) &= \hat{\underline{x}}(t+1/t)\\ &+ \underline{H}\left[\underline{z}(t+1) - \underline{C}\hat{\underline{x}}(t+1/t)\right]\end{aligned}} \quad (31)$$

19

- Combine predict and update equations

$$\boxed{\begin{aligned}\underline{\hat{x}}(t+1/t+1) = &\left[\underline{A} - \underline{H}\,\underline{C}\,\underline{A}\right]\underline{\hat{x}}(t/t)\\ &+ \left[\underline{B} - \underline{H}\,\underline{C}\,\underline{B}\right]\underline{u}(t)\\ &+ \underline{H}\,\underline{z}(t+1)\end{aligned}} \quad (32)$$

- The Kalman filter system matrix $\left[\underline{A} - \underline{H}\,\underline{C}\,\underline{A}\right]$ has all eigenvalues less than unity

A can be unstable!

- Hence, steady state Kalman filter is asymptotically stable in the large

← consequence of controllability and observability assumptions

20

SPECIAL CASE NO PLANT NOISE

violates controllability assumption!

- If

$$\underline{\xi}(t) = \underline{0} \quad \left(\text{set } \underline{\Xi} = \underline{0} \text{ or } \underline{L} = \underline{0}\right) \quad (33)$$

MUST USE TIME-VARYING KF!

then

$$\boxed{\begin{aligned}\underline{\Sigma}(t+1/t+1) = \underline{A}\Big[&\underline{\Sigma}(t/t) - \underline{\Sigma}(t/t)\underline{A}'\underline{C}'\cdot\\ &\left(\underline{C}\,\underline{A}\,\underline{\Sigma}(t/t)\underline{A}'\underline{C}' + \underline{\Theta}\right)^{-1}\underline{C}\,\underline{A}\,\underline{\Sigma}(t/t)\Big]\underline{A}'\end{aligned}} \quad (34)$$

21

- Under observability assumptions

$$\lim_{t\to\infty} \underline{\Sigma}(t/t) = \underline{\Sigma} = \underline{0} \quad (35)$$

$$\lim_{t\to\infty} \underline{\hat{x}}(t/t) = \underline{x}(t) \quad (36)$$

Hence, perfect state estimation occurs

with many many measurements, we remove all state uncertainty!

22

- Note that

$$\lim_{t\to\infty} \underline{H}(t) = \underline{0} \quad (37)$$

i.e. asymptotically Kalman filter runs "open-loop"

- It makes no sense to implement steady-state Kalman filter

23

- To compensate for uncertainties in $\underline{A}$, $\underline{B}$, $\underline{L}$, $\underline{C}$ one must use a non-zero value for $\underline{\Xi}$
- This tends to prevent divergence of Kalman filter.

24

SUMMARY OF CALCULATIONS

- Off-Line Computations

Step 1: Determine unique positive definite solution matrix $\underline{\Sigma}_p$ of algebraic matrix equation - see eq. (23)

$$\underline{0} = -\underline{\Sigma}_p + \underline{A}\,\underline{\Sigma}_p\,\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}' - \underline{A}'\,\underline{\Sigma}_p\,\underline{C}' \left[\underline{C}\,\underline{\Sigma}_p\,\underline{C}' + \underline{\Theta}\right]^{-1} \underline{C}\,\underline{\Sigma}_p\,\underline{A} \qquad (38)$$

$$\underline{\Sigma}_p = \lim_{t \to \infty} \underline{\Sigma}(t+1/t) \qquad (39)$$

25

Step 2:(Optional) Find

$$\underline{\Sigma} = \underline{\Sigma}_p - \underline{\Sigma}_p\,\underline{C}' \left[\underline{C}\,\underline{\Sigma}_p\,\underline{C}' + \underline{\Theta}\right]^{-1} \underline{C}\,\underline{\Sigma}_p \qquad (40)$$

$$\underline{\Sigma} = \lim_{t \to \infty} \underline{\Sigma}(t/t) \qquad (41)$$

- $\underline{\Sigma}$ is useful to deduce RMS state estimation accuracy a priori

26

Step 3: Calculate steady state filter gain matrix $\underline{H}$

$$\underline{H} = \underline{\Sigma}_p\,\underline{C}' \left[\underline{C}\,\underline{\Sigma}_p\,\underline{C}' + \underline{\Theta}\right]^{-1} \qquad (42)$$

or

$$\underline{H} = \underline{\Sigma}\,\underline{C}'\,\underline{\Theta}^{-1} \qquad (43)$$

27

- On-Line Calculations

Step 4: Construct state estimate

$$\boxed{\begin{aligned}\hat{\underline{x}}(t+1/t+1) &= [\underline{A} - \underline{H}\underline{C}\underline{A}]\ \hat{\underline{x}}(t/t)\\ &+ [\underline{B} - \underline{H}\underline{C}\underline{B}]\ \underline{u}(t) + \underline{H}\underline{z}(t+1)\end{aligned}} \quad (44)$$

$$\hat{\underline{x}}(0/0) = E\{\underline{x}(0)\} \quad (45)$$

28

Note: Residual sequence (at steady-state!)

$$\underline{r}(t+1) \triangleq \underline{z}(t+1) - \underline{C}\hat{\underline{x}}(t+1/t) \quad (46)$$

$$= \underline{z}(t+1) - \underline{C}[\underline{A}\hat{\underline{x}}(t/t) + \underline{B}\underline{u}(t)]$$

is stationary, gaussian, zero mean white with

$$\boxed{\text{cov}[\underline{r}(t); \underline{r}(\tau)] = [\underline{C}\Sigma_p\underline{C}' + \underline{\Theta}]\delta_{t\tau}} \quad (47)$$

↑

Finite residual covariance; uncorrelated in time!