REF. NO. KF #11

1

STOCHASTIC ESTIMATION

Numerical Example:

Estimation of positions, velocities, and accelerations

2

MOTIVATION

- Understand structure of Kalman-Bucy filter
- Illustrate asymptotic properties of covariances and filter gains

3

PHYSICAL PROBLEM

- Motion of mass in frictionless environment subject to

- white noise acceleration input
- white noise jerk input

4

- We measure its position only in the presence of additive white noise
- We want to estimate its position, velocity and acceleration

5

PHYSICAL STATE VARIABLES

$x_1(t)$ = position

$x_2(t)$ = velocity

$x_3(t)$ = acceleration

6

• DYNAMICS

$$\left.\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t) + \xi_2(t)\\ \dot{x}_3(t) &= \xi_3(t)\end{aligned}\right\} \quad (1)$$

$\xi_2(t)$: white acceleration noise

$\xi_3(t)$: white jerk noise

• MEASUREMENT

$$z(t) = x_1(t) + \theta(t) \quad (2)$$

= Noisy position measurement

7

STATE DYNAMICS:

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) + \underline{\xi}(t) \quad (3)$$

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{B} = \underline{0} \quad (4)$$

$$\underline{\xi}(t) = \begin{bmatrix} 0 \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix}$$

8

MEASUREMENT EQUATION:

$$\underline{z}(t) = \underline{C}\,\underline{x}(t) + \underline{\theta}(t) \quad (5)$$

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad (6)$$

9
PROBABILISTIC INFORMATION

$$\text{cov}\left[\underline{x}(0); \underline{x}(0)\right] = \underline{\Sigma}_0 = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (7)$$

$$\text{cov}\left[\underline{\xi}(t); \underline{\xi}(\tau)\right] = \underline{\Xi}\,\delta(t-\tau); \quad (8)$$

$$\underline{\Xi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 20 \end{bmatrix} \quad (9)$$

$$\text{cov}\left[\underline{\theta}(t); \underline{\theta}(\tau)\right] = \underline{\Theta}\,\delta(t-\tau) \quad (10)$$

$$\Theta = 1 \quad \text{(scalar)} \quad (11)$$

10
ERROR COVARIANCE MATRIX: $\underline{\Sigma}(t)$ (3 x 3)

$$\underline{\Sigma}(t) = \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \sigma_{13}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) & \sigma_{23}(t) \\ \sigma_{13}(t) & \sigma_{23}(t) & \sigma_{33}(t) \end{bmatrix} \quad (12)$$

$$\dot{\underline{\Sigma}}(t) = \underline{A}\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\,\underline{A}' + \underline{\Xi} - \underline{\Sigma}(t)\,\underline{C}'\,\underline{\Theta}^{-1}\,\underline{C}\,\underline{\Sigma}(t); \quad \underline{\Sigma}(0) = \underline{\Sigma}_0 \quad (13)$$

11
FILTER GAIN MATRIX $\underline{H}(t)$ (3 x 1)

$$\underline{H}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} \quad (14)$$

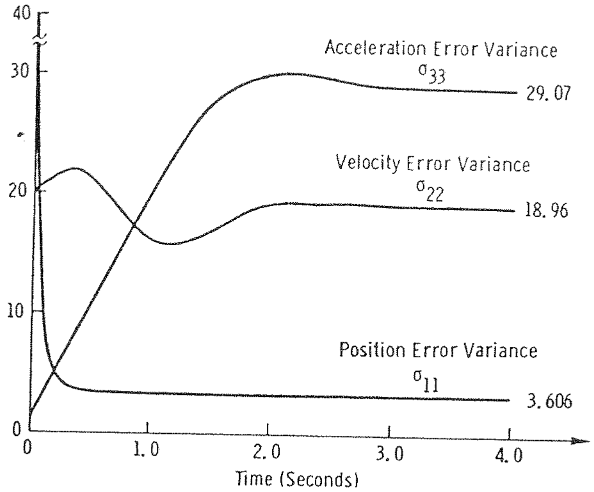
$$\underline{H}(t) = \underline{\Sigma}(t)\,\underline{C}'\,\underline{\Theta}^{-1} \quad (15)$$

$$\left.\begin{aligned} h_1(t) &= \sigma_{11}(t) \\ h_2(t) &= \sigma_{12}(t) \\ h_3(t) &= \sigma_{13}(t) \end{aligned}\right\} \quad (16)$$

12

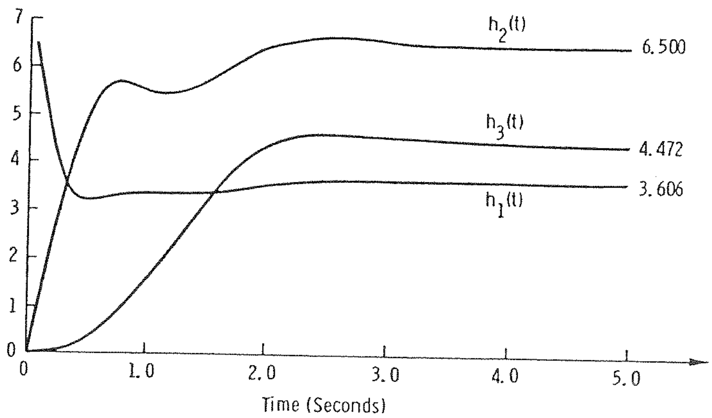
VARIANCES OF STATE ESTIMATION ERRORS

$\sigma_{ii}(t) = E\{(x_i(t) - \hat{x}_i(t))^2\};\ i = 1, 2, 3$

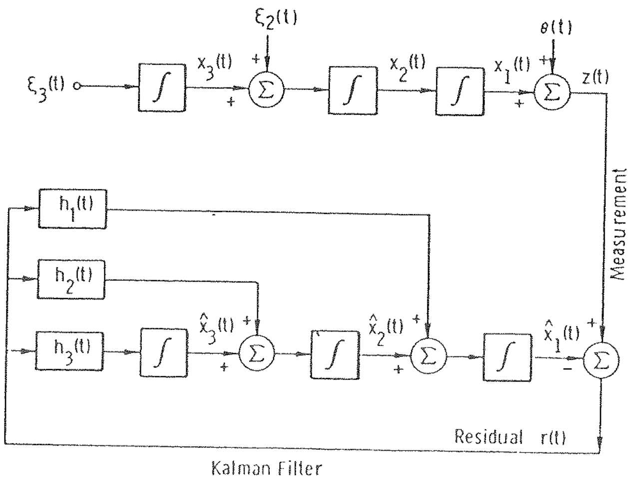


13

KALMAN FILTER GAINS



14



15
STEADY STATE FILTER GAINS

$$\left.\begin{aligned}\lim_{t\to\infty} h_1(t) &= h_1 = 3.606 \\ \lim_{t\to\infty} h_2(t) &= h_2 = 6.500 \\ \lim_{t\to\infty} h_3(t) &= h_3 = 4.472\end{aligned}\right\} \qquad (17)$$

16
STEADY STATE ESTIMATION ACCURACY

As $t\to\infty$ the accuracy of the state estimation error $x_i(t) - \hat{x}_i(t)$, $i = 1,2,3$ can be evaluated by the variance limit

$$\lim_{t\to\infty} \sigma_{ii}(t) = \sigma_{ii} \qquad (18)$$

$$\left.\begin{aligned}\lim_{t\to\infty} \sigma_{11}(t) &= \sigma_{11} = 3.606 \\ \lim_{t\to\infty} \sigma_{22}(t) &= \sigma_{22} = 18.96 \\ \lim_{t\to\infty} \sigma_{33}(t) &= \sigma_{33} = 29.07\end{aligned}\right\} \qquad (19)$$