Correlated and Colored Plant and Measurement Noises

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Last Revised: November 28, 2001

Ref. No. KF #12A

Theme

- We examine two related problems for continuous-time optimal estimation problems that are characterized by different assumptions on the plant and the measurement noises, [1]-[3]
- First, we examine the case that the plant noise $\xi(t)$ and the sensor noise $\theta(t)$ are continuous-time white noise processes which are not independent, i.e. they are "correlated"
- Second, we consider problems for which the plant noise and the measurement noise are not pure white noise processes, i.e. they are "colored"
  - we need the results of the correlated-noise case to derive the results for the colored-noise case
- The basic ideas can be extended to discrete-time filtering problems, [1]-[3]

Part I: Correlated Plant and Sensor White Noise

Basic Problem

- Previous assumption

(1) $E\{\xi(t)\} = 0,\ E\{\xi(t)\xi'(\tau)\} = \Xi(t)\delta(t-\tau)$

(2) $E\{\theta(t)\} = 0,\ E\{\theta(t)\theta'(\tau)\} = \Theta(t)\delta(t-\tau)$

(3) $x(t_0),\ \xi(t),\ \theta(\tau)$ independent $\forall t_0, t, \tau$

(4) $E\{\xi(t)\theta'(\tau)\} = 0$ independent $\forall t, \tau$

- New assumption

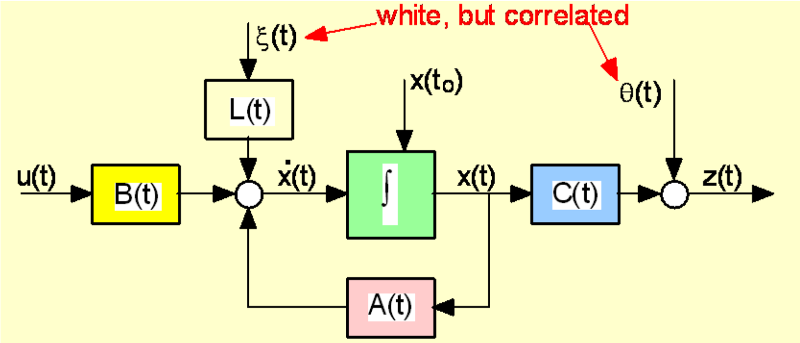
(5) $E\{\xi(t)\} = 0,\ E\{\xi(t)\xi'(\tau)\} = \Xi(t)\delta(t-\tau)$

(6) $E\{\theta(t)\} = 0,\ E\{\theta(t)\theta'(\tau)\} = \Theta(t)\delta(t-\tau)$

(7) $x(t_0)$ independent of

$\xi(t),\ \theta(\tau)\quad \forall t_0, t, \tau$

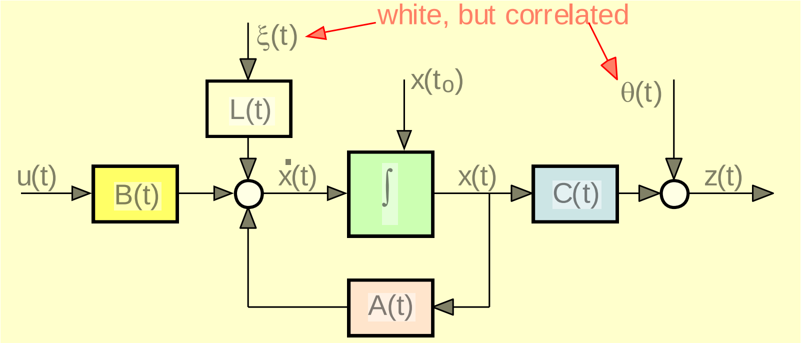
(8) $E\{\xi(t)\theta'(\tau)\} = \Psi(t)\delta(t-\tau)$



PROBLEM

- We seek the optimal state-estimate $\hat{x}(t)$ and its covariance matrix $\Sigma(t)$

Plant and Sensors Modeling



- Plant and sensor modeling: $x(t) \in R^n,\ u(t) \in R^q,\ \xi(t) \in R^p$

(9) $$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + L(t)\xi(t); \quad x(t_o) = x_0$$

(10) $$z(t) = C(t)x(t) + \theta(t); \quad z(t) \in R^m,\ \theta(t) \in R^m$$

- Initial state: $x(t_0) \sim N\left(\bar{x}_0, \Sigma_0\right)$ independent of $\xi(t),\ \theta(\tau) \quad \forall t_0, t, \tau$
- Correlated white noises

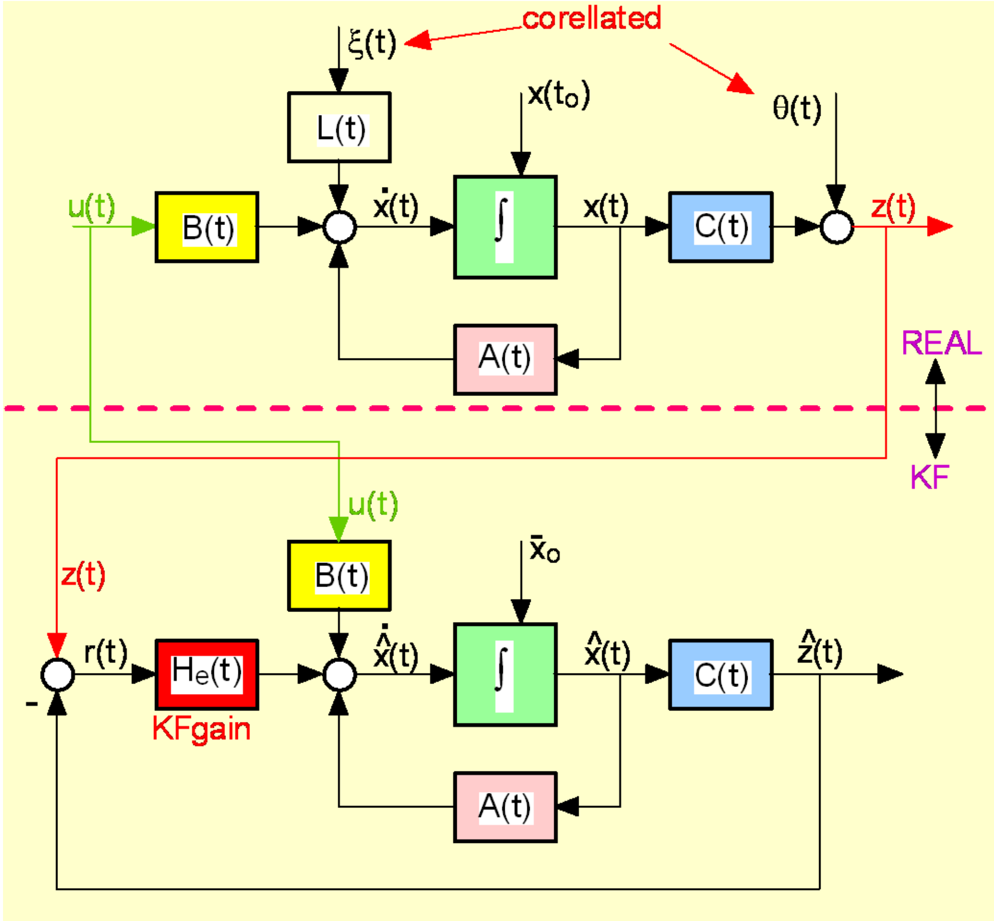
(11) $E\{\xi(t)\} = 0,\ E\{\xi(t)\xi'(\tau)\} = \Xi(t)\delta(t-\tau); \quad \Xi(t)\ p \times p$ matrix

(12) $E\{\theta(t)\} = 0,\ E\{\theta(t)\theta'(\tau)\} = \Theta(t)\delta(t-\tau); \quad \Theta(t)\ m \times m$ matrix

(13) $E\{\xi(t)\theta'(\tau)\} = \Psi(t)\delta(t-\tau); \quad \Psi(t)\ p \times m$ matrix

Structure of Optimal Filter

- The structure of the optimal KF for the corellated noise case is identical to that of the classical KBF, except that the filter gain matrix, $H_e(t)$, is computed by a different formula



Summary of KBF Equations

- State - estimate dynamics

(14) $$\frac{d\hat{x}(t)}{dt} = A(t)\hat{x}(t) + B(t)u(t) + H_e(t)[z(t) - C(t)\hat{x}(t)]; \quad \hat{x}(t_0) = \bar{x}_0$$

- Optimal filter gain matrix for correlated noise case

(15) $$H_e(t) = \left[\Sigma_e(t)C'(t) + L(t)\Psi(t)\right]\Theta^{-1}(t)$$

where $\Sigma_e(t) = \Sigma_e'(t) \geq 0$ is the optimal state - covariance matrix that is the solution of the modified matrix Riccati differential equation

(16) $$\frac{d\Sigma_e(t)}{dt} = \left[A(t) - L(t)\Psi(t)\Theta^{-1}(t)C(t)\right]\cdot\Sigma_e(t) +$$

$$+\Sigma_e(t)\cdot\left[A(t) - L(t)\Psi(t)\Theta^{-1}(t)C(t)\right]' + L(t)\left[\Xi(t) - \Psi(t)\Theta^{-1}(t)\Psi'(t)\right]L'(t)$$

$$-\Sigma_e(t)C'(t)\Theta^{-1}(t)C(t)\Sigma_e(t); \quad \Sigma_e(t_0) = \Sigma_0$$

- Note that when the noises are uncorrelated, $\Psi(t) = 0$, eqs. (14) to (16) reduce to those of the standard KBF

Elements of Proof, I

- The basic idea is to make a change of variables so that the new system can be solved using the standard KBF equations (with uncorellated noises)
- Define a $n \times m$ matrix $D(t)$ as follows:

(17) $$D(t) \equiv L(t)\Psi(t)\Theta^{-1}(t)$$

- From the measurement equation (10) we have

(18) $$0 = z(t) - C(t)x(t) - \theta(t) \quad \Rightarrow \quad 0 = D(t)\left[z(t) - C(t)x(t) - \theta(t)\right]$$

- Add the "special 0" of eq. (18) to the state dynamics (9)

(19) $$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + L(t)\xi(t) + \underbrace{D(t)\left[z(t) - C(t)x(t) - \theta(t)\right]}_{0} \Rightarrow$$

(20) $$\frac{dx(t)}{dt} = \left[A(t) - D(t)C(t)\right]x(t) + B(t)u(t) + D(t)z(t) + \\ + \underbrace{\left[L(t)\xi(t) - D(t)\theta(t)\right]}_{\xi_e(t)}$$

Elements of Proof, II

- Using the new random process $\xi_e(t)$ defined by

(21) $$\xi_e(t) \equiv L(t)\xi(t) - D(t)\theta(t) = L(t)\left[\xi(t) - \Psi(t)\Theta^{-1}(t)\theta(t)\right]$$

in the state dynamics (20) and sensor equation (10) we obtain

(22) $$\frac{dx(t)}{dt} = [A(t) - D(t)C(t)]x(t) + B(t)u(t) + D(t)z(t) + \xi_e(t)$$

(23) $$z(t) = C(t)x(t) + \theta(t)$$

- In eq. (22) the term $B(t)u(t) + D(t)z(t)$ represent known time - functions. If we could prove that $\xi_e(t)$ is zero - mean continuous white noise AND is uncorellated with $\theta(t)$ we can then apply the standard KBF theory to the system of eqs. (22) and (23)

Elements of Proof, III

- From eq. (21) $\xi_e(t) = L(t)\xi(t) - D(t)\theta(t),$ so that

(24) $$E\{\xi_e(t)\} = L(t)\underbrace{E\{\xi(t)\}}_{0} - D(t)\underbrace{E\{\theta(t)\}}_{0} = 0$$

(25) $$E\{\xi_e(t)\xi_e'(\tau)\} = E\left\{[L(t)\xi(t) - D(t)\theta(t)][L(\tau)\xi(\tau) - D(\tau)\theta(\tau)]'\right\} =$$

$$= L(t)\underbrace{E\{\xi(t)\xi'(\tau)\}}_{\Xi(t)\delta(t-\tau)}L'(\tau) + D(t)\underbrace{E\{\theta(t)\theta'(\tau)\}}_{\Theta(t)\delta(t-\tau)}D'(\tau)$$

$$- D(t)\underbrace{E\{\theta(t)\xi'(\tau)\}}_{\Psi'(t)\delta(t-\tau)}L'(\tau) - L(t)\underbrace{E\{\xi(t)\theta'(\tau)\}}_{\Psi(t)\delta(t-\tau)}D'(\tau) \quad \Rightarrow$$

(26) $$E\{\xi_e(t)\xi_e'(\tau)\} = [L(t)\Xi(t)L'(t) + D(t)\Theta(t)D'(t)]\delta(t-\tau)$$

$$-[D(t)\Psi'(t)L'(t) + L(t)\Psi(t)D'(t)]\delta(t-\tau)$$

- Clearly, $\xi_e(t)$ is zero-mean continuous white noise with covariance

(27) $$E\{\xi_e(t)\xi_e'(\tau)\} = \Xi_e(t)\delta(t-\tau)$$

(28) $$\Xi_e(t) = L(t)\Xi(t)L'(t) + D(t)\Theta(t)D'(t) - D(t)\Psi'(t)L'(t) - L(t)\Psi(t)D'(t)$$

$$= L(t)\left[\Xi(t) - \Psi(t)\Theta^{-1}(t)\Psi'(t)\right]L'(t)$$

Elements of Proof, IV

- Next, we show that $\xi_e(t)$ and $\theta(t)$ are uncorrelated, i.e. that

(29) $$E\{\xi_e(t)\theta'(\tau)\} = 0 \quad \forall t, \tau$$

- From eq. (21) we have

(30) $$E\{\xi_e(t)\theta'(\tau)\} = E\left\{\left[L(t)\xi(t) - L(t)\Psi(t)\Theta^{-1}(t)\theta(t)\right]\cdot\theta'(\tau)\right\} =$$

$$= L(t)\underbrace{E\{\xi(t)\theta'(\tau)\}}_{\Psi(t)\delta(t-\tau)} - L(t)\Psi(t)\Theta^{-1}(t)\underbrace{E\{\theta(t)\theta'(\tau)\}}_{\Theta(t)\delta(t-\tau)} =$$

$$= L(t)\Psi(t)\delta(t-\tau) - L(t)\Psi(t)\delta(t-\tau) = 0 \qquad QED$$

- Since we have established that $\xi_e(t)$ and $\theta(t)$ are uncorrelated, we can apply the standard KBF theory to the system of equations (22) and (23) and verify that the optimal filter is that described by eqs. (14), (15) and (16). QED

Remark

- The correlation matrix $\Psi(t)$ cannot be arbitrary. From the definitions of eqs. (11) to (13) we compute

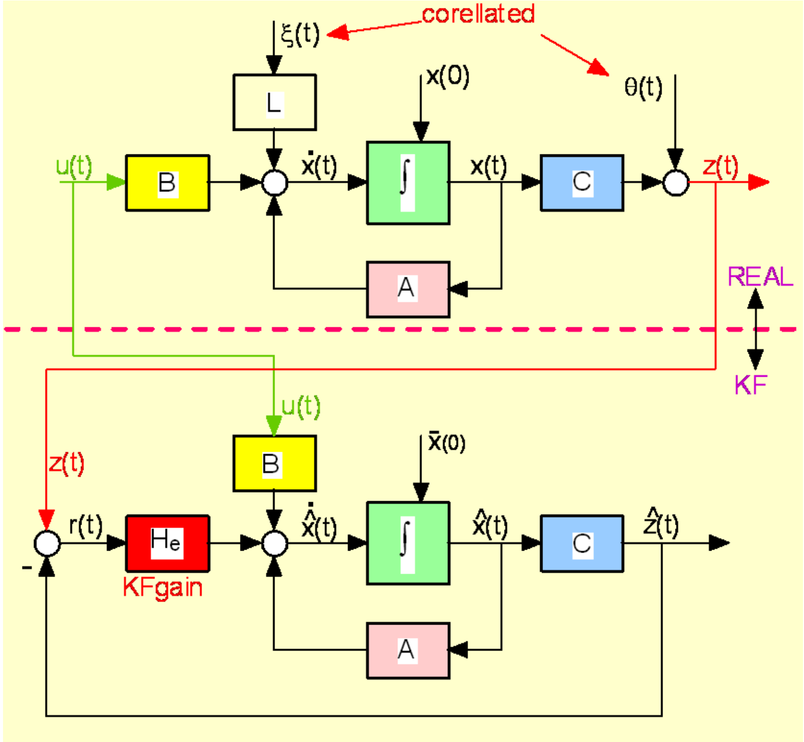
(31)
$$E\left\{\begin{bmatrix}\xi(t)\\ \theta(t)\end{bmatrix}\cdot\begin{bmatrix}\xi(\tau)\\ \theta(\tau)\end{bmatrix}'\right\}=\begin{bmatrix}E\{\xi(t)\xi'(\tau)\} & E\{\xi(t)\theta'(\tau)\}\\ E\{\theta(t)\xi'(\tau)\} & E\{\theta(t)\theta'(\tau)\}\end{bmatrix}=$$
$$=\underbrace{\begin{bmatrix}\Xi(t) & \Psi(t)\\ \Psi'(t) & \Theta(t)\end{bmatrix}}_{Q(t)}\cdot\delta(t-\tau)$$

- Since $Q(t)$ is a covariance matrix, it must be symmetric (which it is) and positive-semidefinite. This positive-semidefinite requirement constrains the correlation matrix $\Psi(t)$.
- Also remember that $\Theta(t)$ must also be positive definite, so that $\Theta^{-1}(t)$ exists

The Steady-State Case

ASSUMPTIONS

- All matrices in the plant and sensor equations are constant
- The covariance matrices are time - invariant
- The pair [A, L] is stabilizable (or controllable)
- The pair [A, C] is detectable (or observable)
- THEN, the steady - state KBF is a linear, time - invariant system, and the KF gain matrix H_e is constant



The Steady-State Case: Summary

- Steady-state KBF equations for the correlated noise case

(32) $$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + H_e[z(t) - C\hat{x}(t)]; \quad \hat{x}(0) = E\{x(0)\}$$

- The constant $n \times m$ filter gain matrix H_e is given by

(33) $$H_e = [\Sigma_e C' + L\Psi]\Theta^{-1}$$

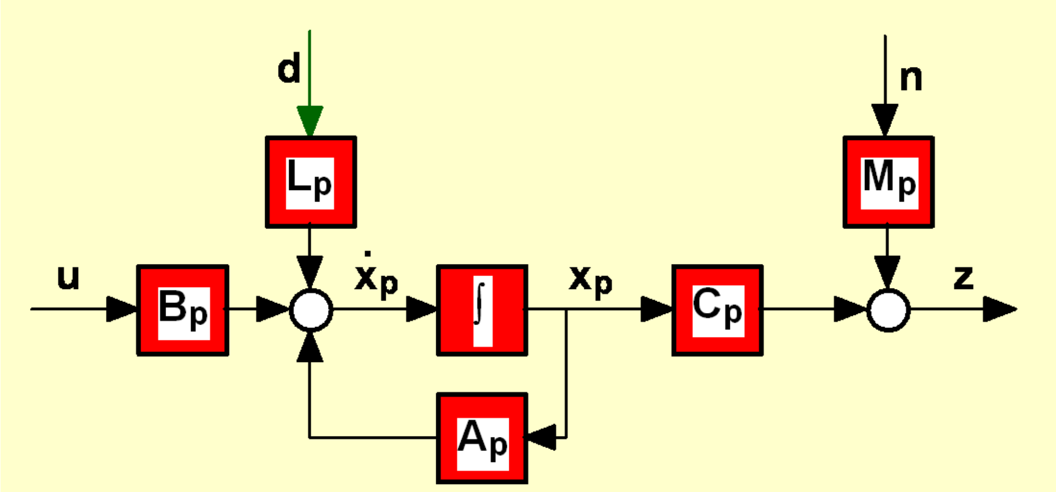
where the $n \times n$ symmetric and positive semidefinite matrix Σ_e is the unique solution matrix of the modified Riccati algebraic equation

(34) $$0 = \left[A - L\Psi\Theta^{-1}C\right]\Sigma_e + \Sigma_e\left[A - L\Psi\Theta^{-1}C\right]' + L\left[\Xi - \Psi\Theta^{-1}\Psi'\right]L' - \Sigma_e C'\Theta^{-1}C\Sigma_e$$

- The stabilizability assumption of [A, B] and the detectability assumption of [A, C] guarantee the asymptotic stability of the filter in eq. (32)

Part II: Colored Plant and Sensor Noise

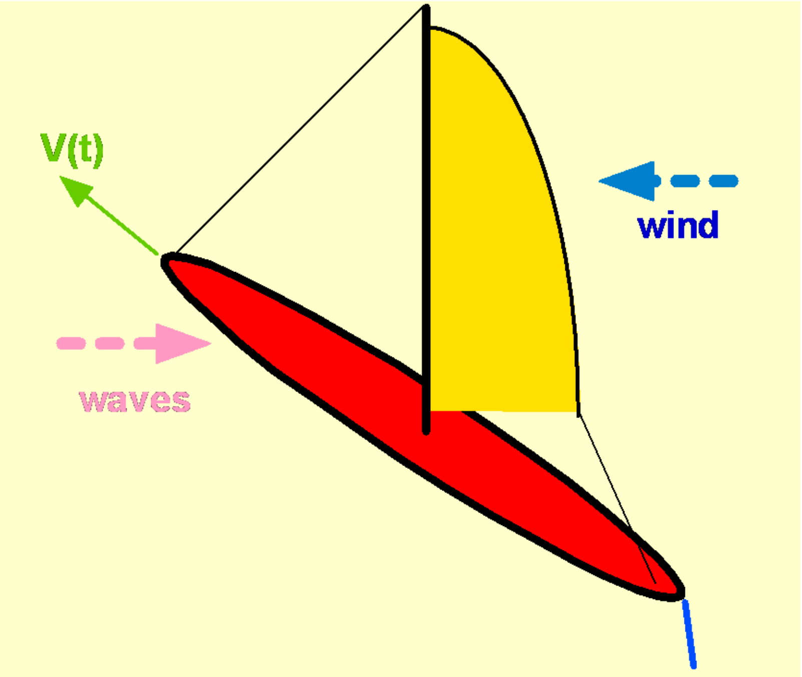
The Basic Issue



- In many practical applications, the physical disturbances acting on the plant are NOT white
- In general, we expect the plant disturbance vector, *d(t),* to be a colored noise process
- Also, the sensor noise vector, *n(t),* can often contain colored noise elements
- We need to extend the optimal estimation framework to handle colored noise inputs

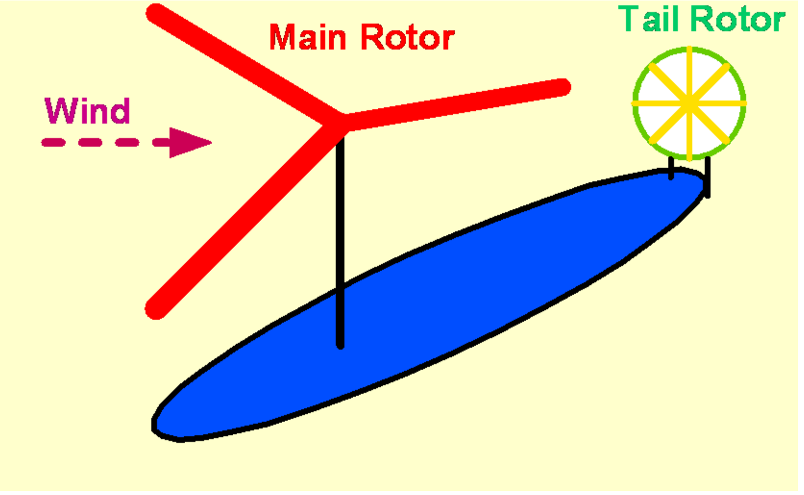
Sailboat Example

- For a sailboat the physical disturbances correspond to the forces and moments generated by the wind and the waves
- It is completely unrealistic to model the wind and wave disturbances as white noise
- But, using the concept of “prewhitening” we can model them as the outputs of (fictitious) dynamical systems driven by (fictitious) white noise



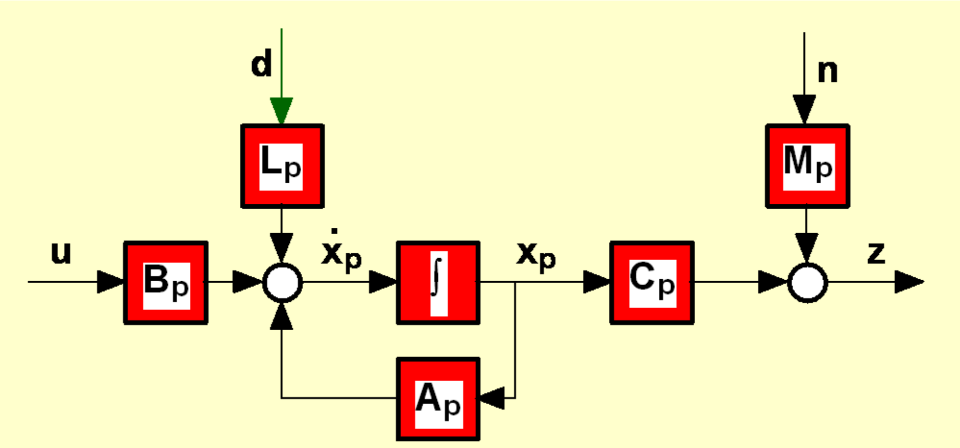
Helicopter Example

- The physical sensors (gyros and accelerometers) that are used to measure the attitude of a helicopter will contain "colored noise" due to the vibrations caused by the rotation of the main and tail rotors, in addition to a white noise component
- Also, the wind disturbances will be colored noise
- Once more, we can use the "prewhitening" concept for both disturbance and sensor noise



Plant Dynamics

- Both the plant disturbance $d(t)$ and the sensor noise $n(t)$ are colored noise random processes
- The sensor noise $n(t)$ MUST contain a white noise component ($d(t)$ may as well)



- Plant dynamics: $x_p(t) \in R^{n_p};\ u(t) \in R^{q_p};\ d(t) \in R^{p_p}$

(35) $$\frac{dx_p(t)}{dt} = A_p(t)x_p(t) + B_p(t)u(t) + L_p(t)d(t); \quad x_p(t_0) \sim N\left(x_{p0}, \Sigma_{p0}\right)$$

- Measurement equation: $z(t) \in R^{m_p};\ n(t) \in R^{m_p}$

(36) $$z(t) = C_p(t)x_p(t) + M_p(t)n(t); \quad M_p^{-1}(t) \text{ exists}$$

The Basic Idea

- We have developed the Kalman-Bucy filter (KBF) equations under the assumptions that both the plant noise and the measurement noise are "pure white" random processes
- In this version of the problem we allow the plant noise *d(t)* and the sensor noise *n(t)* to also contain some "colored noise" elements
- The key idea is to transform the "colored noise" version into the standard formulation so that we can apply the KBF equations
  - this is done by using the concept of "prewhitening", extended to the time-varying case, so that we represent the colored disturbance *d(t)* and the colored sensor noise *n(t)* as the output of fictitious LTV dynamic systems driven by pure white noise
- **Caution:** the sensor noise *n(t)* must contain a (possibly small) pure white-noise component for the KBF to be applicable

Disturbance Dynamics

- The disturbance vector $d(t)$ that drives the plant dynamics is modeled as the output of an $n_d -$dimensional dynamic system driven by gaussian zero mean, unit - intensity pure white noise

(37) $$\frac{dx_d(t)}{dt} = A_d(t)x_d(t) + B_d(t)\xi_d(t)$$

(38) $$d(t) = C_d(t)x_d(t) + D_d(t)\xi_d(t)$$

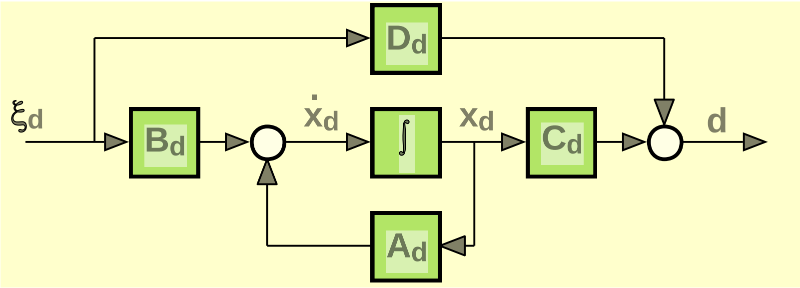
(39) $$E\{\xi_d(t)\} = 0; \quad cov[\xi_d(t); \xi_d(\tau)] = I \cdot \delta(t-\tau)$$

(40) $$x_d(t_0) \sim N(\bar{x}_{d0}, \Sigma_{d0})$$

- The vector dimensions are

(41) $x_d(t) \in R^{n_d}; \xi_d(t) \in R^{p_d}; d(t) \in R^{p_p}$

- If $D_d(t) = 0,$ then the plant disturbance vector $d(t)$ has no white noise component



Sensor Noise Dynamics

- The sensor noise $n(t)$ is modeled as the output of an n_s – dimensional LTV dynamic system driven by a gaussian zero-mean, unit-intensity white noise $\theta_s(t)$

(42) $$\frac{dx_s(t)}{dt} = A_s(t)x_s(t) + B_s(t)\theta_s(t)$$

(43) $$n(t) = C_s(t)x_s(t) + D_s(t)\theta_s(t)$$

(44) $$E\{\theta_s(t)\} = 0, \quad cov[\theta_s(t); \theta_s(\tau)] = I \cdot \delta(t - \tau)$$

(45) $$x_s(t) \sim N(\bar{x}_{s0}, \Sigma_{s0})$$

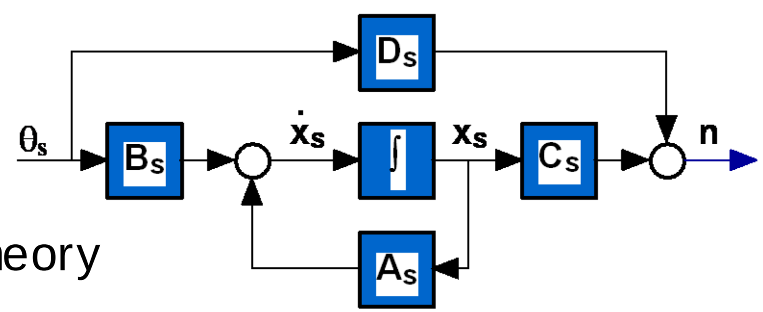
- The vector dimensions are

(46) $$x_s(t) \in R^{n_s}, \theta_s(t) \in R^{m_p}, n(t) \in R^{m_p}$$

- We MUST assume that

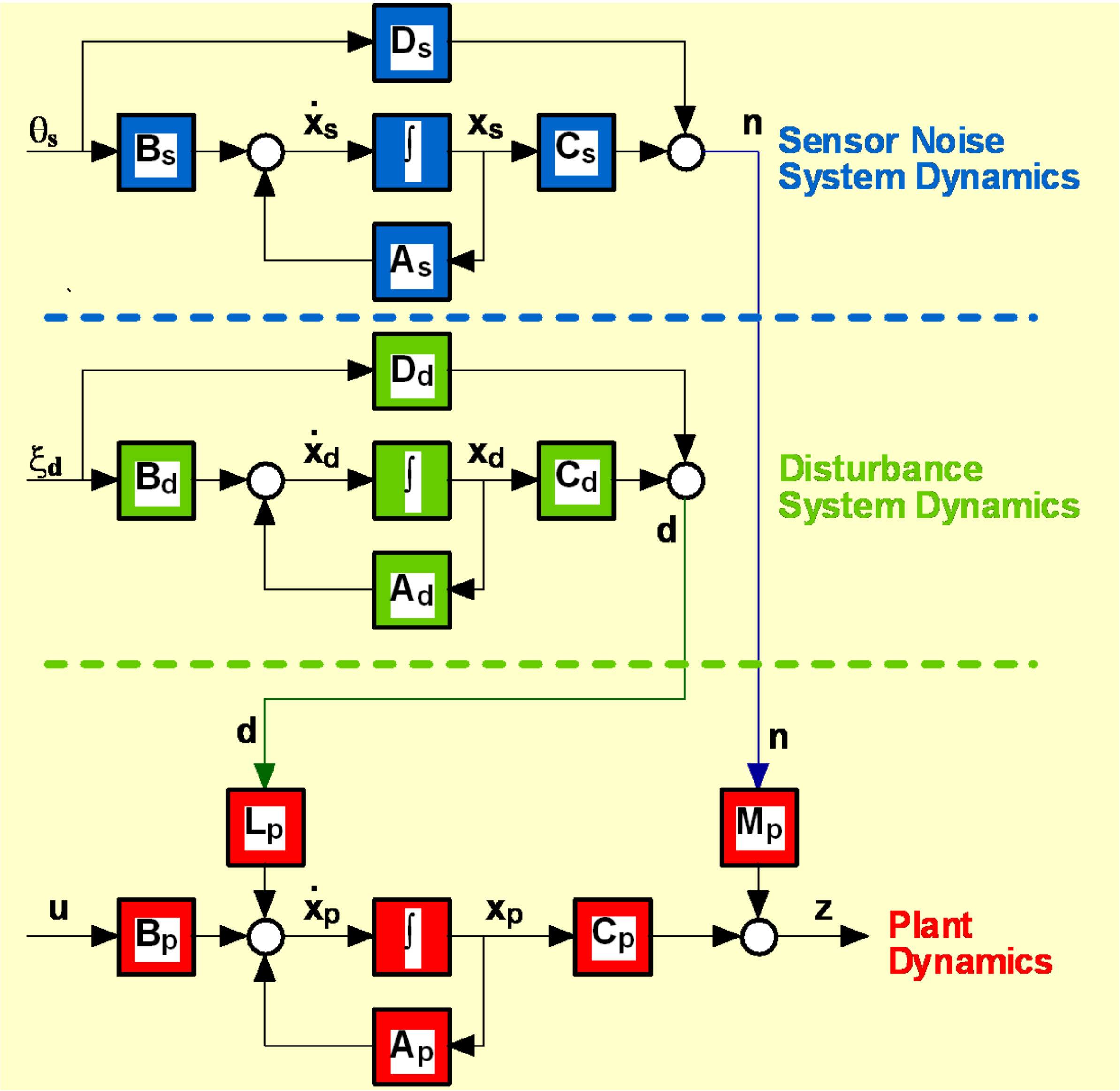
(47) $D_s^{-1}(t)$ exists

otherwise, we cannot use the standard KBF theory



- Eq. (47) implies that the sensor noise always contains a white noise component

Visualization of Augmented System



Augmented System Equations

- Combine the plant, disturbance and sensor dynamics in an augmented state-space system

(48) $$\frac{d}{dt}\underbrace{\begin{bmatrix} x_p(t) \\ x_d(t) \\ x_s(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} A_p(t) & L_p(t)C_d(t) & 0 \\ 0 & A_d(t) & 0 \\ 0 & 0 & A_s(t) \end{bmatrix}}_{A(t)} \cdot \underbrace{\begin{bmatrix} x_p(t) \\ x_d(t) \\ x_s(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} B_p(t) \\ 0 \\ 0 \end{bmatrix}}_{B(t)} u(t) + $$

$$+ \underbrace{\begin{bmatrix} L_p(t)D_d(t) & 0 \\ B_d(t) & 0 \\ 0 & B_s(t) \end{bmatrix}}_{L(t)} \cdot \underbrace{\begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix}}_{\xi(t)}$$

(49) $$z(t) = \underbrace{\begin{bmatrix} C_p(t) & 0 & M_p(t)C_s(t) \end{bmatrix}}_{C(t)} \cdot \underbrace{\begin{bmatrix} x_p(t) \\ x_d(t) \\ x_s(t) \end{bmatrix}}_{x(t)} + \underbrace{M_p(t)D_s(t)\theta_s(t)}_{\theta(t)}$$

- The system (48) and (49) is in the standard KBF form, except that the white noises $\xi(t)$ and $\theta(t)$ are correlated (as in Part I)

Correlation Matrix

- Assume that the disturbance - dynamics white noise $\xi_d(t)$ and the sensor - noise - dynamics white noise $\theta_s(\tau)$ are mutually independent and are also independent of all initial state vectors
- From eqs. (48) and (49) we have

(50) $$\xi(t) \equiv \begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix}; \quad \theta(t) \equiv M_p(t) D_s(t) \theta_s(t)$$

therefore, we can calculate the correlation matrix $\Psi(t)$ by

(51) $$\Psi(t)\delta(t-\tau) \equiv E\left\{\xi(t)\theta'(\tau)\right\} = E\left\{\begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix} \cdot \theta_s'(\tau) D_s'(\tau) M_p'(\tau)\right\} =$$

$$= \begin{bmatrix} \underbrace{E\left\{\xi_d(t)\theta_s'(\tau)\right\}}_{0} \cdot D_s'(\tau) M_p'(\tau) \\ \underbrace{E\left\{\theta_s(t)\theta_s'(\tau)\right\}}_{I\cdot\delta(t-\tau)} D_s'(\tau) M_p'(\tau) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ D_s'(t) M_p'(t) \end{bmatrix}}_{\Psi(t)} \cdot \delta(t-\tau)$$

Covariance Matrices

- It remains to calculate the covariance matrices of the augmented white noises $\xi(t)$ and $\theta(t)$. From eqs. (48) and (49) we have

(52) $$\xi(t) \equiv \begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix}; \quad \theta(t) \equiv M_p(t) D_s(t) \theta_s(t) \quad \Rightarrow$$

(53) $$\Xi(t)\delta(t-\tau) \equiv E\{\xi(t)\xi'(\tau)\} = E\left\{\begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix} \cdot \begin{bmatrix} \xi'_d(\tau) & \theta'_s(\tau) \end{bmatrix}\right\} =$$

$$= \begin{bmatrix} \underbrace{E\{\xi_d(t)\xi'_d(\tau)\}}_{I.\delta(t-\tau)} & \underbrace{E\{\xi_d(t)\theta'_s(\tau)\}}_{0} \\ \underbrace{E\{\theta_s(t)\xi'_d(\tau)\}}_{0} & \underbrace{E\{\theta_s(t)\theta'_s(\tau)\}}_{I.\delta(t-\tau)} \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}}_{\Xi(t)} .\delta(t-\tau)$$

(54) $$\Theta(t)\delta(t-\tau) \equiv E\{\theta(t)\theta'(\tau)\} = M_p(t) D_s(t) \underbrace{E\{\theta_s(t)\theta'_s(\tau)\}}_{I.\delta(t-\tau)} D'_s(\tau) M'_p(\tau) =$$

$$= \underbrace{M_p(t) D_s(t) D'_s(t) M'_p(t)}_{\Theta(t)} .\delta(t-\tau)$$

The Augmented System Equations

- The augmented equations (48) and (49) have the standard form

(55) $$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + L(t)\xi(t)$$

(56) $$z(t) = C(t)x(t) + \theta(t)$$

with the following definitions

(57) $$x(t) \equiv \begin{bmatrix} x_p(t) \\ x_d(t) \\ x_s(t) \end{bmatrix};\ A(t) \equiv \begin{bmatrix} A_p(t) & L_p(t)C_d(t) & 0 \\ 0 & A_d(t) & 0 \\ 0 & 0 & A_s(t) \end{bmatrix};\ B(t) \equiv \begin{bmatrix} B_p(t) \\ 0 \\ 0 \end{bmatrix}$$

$$C(t) \equiv \begin{bmatrix} C_p(t) & 0 & M_p(t)C_s(t) \end{bmatrix};\ \xi(t) \equiv \begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix};\ \theta(t) \equiv M_p(t)D_s(t)\theta_s(t)$$

$$\Xi(t) \equiv \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix};\ \Psi(t) \equiv \begin{bmatrix} 0 \\ D_s'(t)M_p'(t) \end{bmatrix};\ \Theta(t) \equiv M_p(t)D_s(t)D_s'(t)M_p'(t)$$

- We can now apply the equations (14) to (16) to the above system (55) to (57) to obtain the filter equations

The Augmented Covariance and Filter Gain Matrices

- The augmented state vector $x(t)$ in eq. (57) has dimension $n_p + n_d + n_s$
- Thus, the covariance matrix $\Sigma_e(t)$ defined by the Riccati equation (16) is an $(n_p + n_d + n_s) \times (n_p + n_d + n_s)$ matrix, decomposable into

(58)
$$\Sigma_e(t) = \begin{bmatrix} \Sigma_{pp}(t) & \Sigma_{pd}(t) & \Sigma_{ps}(t) \\ \Sigma'_{pd}(t) & \Sigma_{dd}(t) & \Sigma_{ds}(t) \\ \Sigma'_{ps}(t) & \Sigma'_{ds}(t) & \Sigma_{ss}(t) \end{bmatrix}$$

and the filter gain matrix $H_e(t)$ defined by eq. (15) can be decomposed

(59)
$$H_e(t) = \begin{bmatrix} H_p(t) \\ H_d(t) \\ H_s(t) \end{bmatrix} \quad \begin{matrix} n_p \times m_p \text{ matrix} \\ n_d \times m_p \text{ matrix} \\ n_s \times m_p \text{ matrix} \end{matrix}$$

- Given the augmented variable definitions of eq. (57) we can solve the Riccati equation (16) to calculate $\Sigma_e(t)$ and substitute into eq. (15) to calculate the gain matrix $H_e(t)$ and its decomposition (59)

The Filter Equations

- Once we have calculated the filter gain matrix (59) we can state the state-estimate equations of eq. (14), using the decomposition

(60) $$\hat{x}(t) = \begin{bmatrix} \hat{x}_p(t) \\ \hat{x}_d(t) \\ \hat{x}_s(t) \end{bmatrix} \begin{matrix} \leftarrow plant-state\,estimate \\ \leftarrow disturbance-state\,estimate \\ \leftarrow sensor-state\,estimate \end{matrix}$$

- Define the m_p-dimensional residual vector $r(t)$

(61) $$r(t) \equiv z(t) - C_p(t)\hat{x}_p(t) - M_p(t)\underbrace{C_s(t)\hat{x}_s(t)}_{\hat{n}(t)}$$

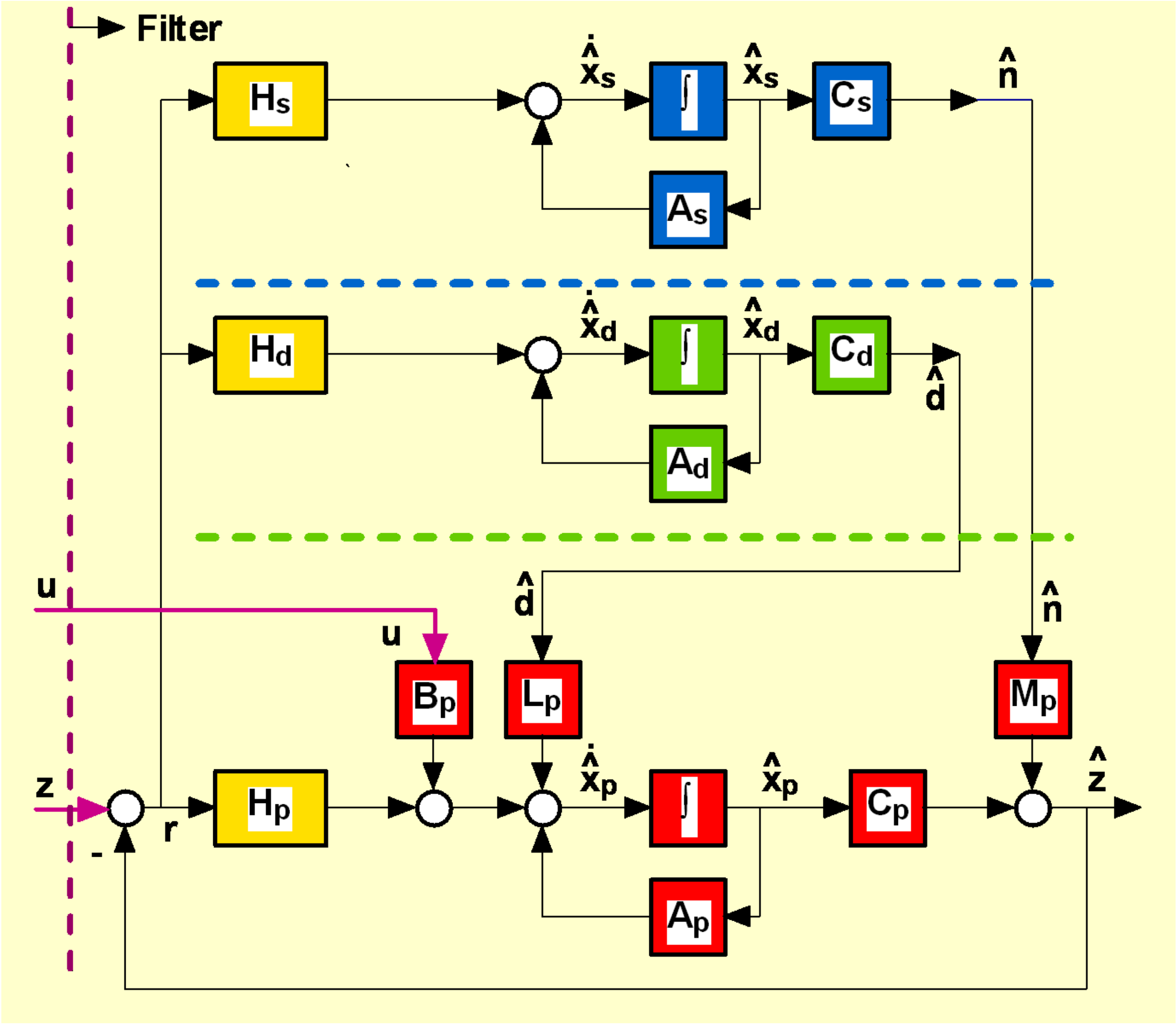
- Then, the different state-estimates are generated by

(62) $$\frac{d\hat{x}_p(t)}{dt} = A_p(t)\hat{x}_p(t) + B_p(t)u(t) + L_p(t)\underbrace{C_d(t)\hat{x}_d(t)}_{\hat{d}(t)} + H_p(t)r(t)$$

(63) $$\frac{d\hat{x}_d(t)}{dt} = A_d(t)\hat{x}_d(t) + H_d(t)r(t)$$

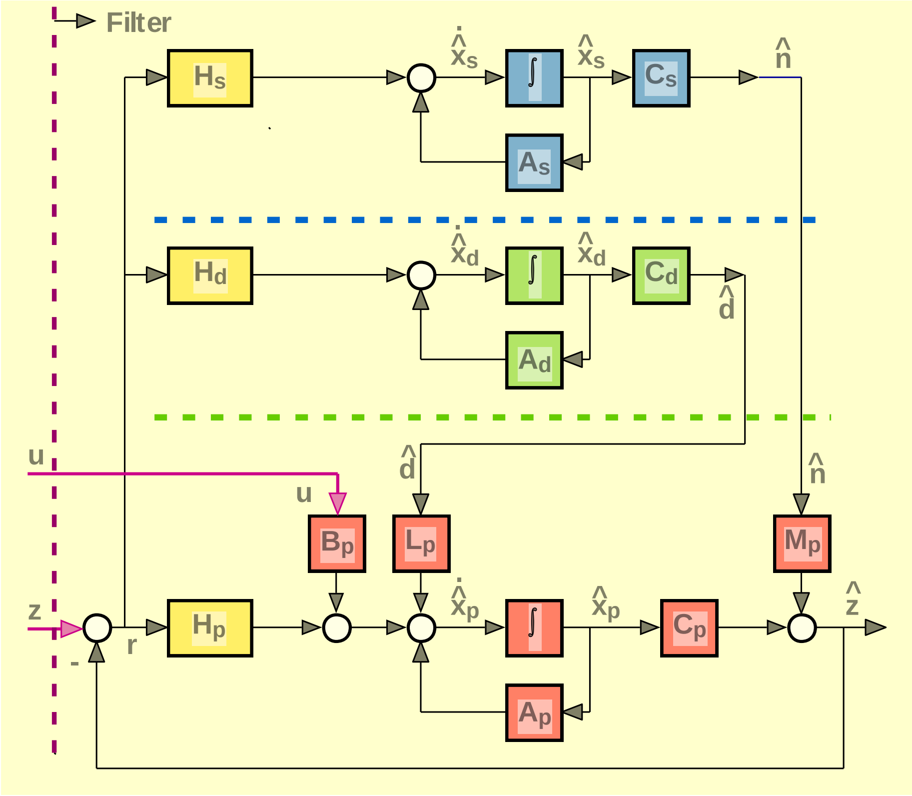
(64) $$\frac{d\hat{x}_s(t)}{dt} = A_s(t)\hat{x}_s(t) + H_s(t)r(t)$$

The Filter Equations: Visualization



Discussion

- It is important to realize that the optimal filter generates an estimate $\hat{d}(t)$ of the colored-part of the plant disturbance $d(t)$ using the disturbance state-estimate $\hat{x}_d(t)$
- Similarly, the optimal filter generates an estimate $\hat{n}(t)$ of the colored-part of the sensor noise $n(t)$ using the sensor dynamics state-estimate $\hat{x}_s(t)$



The residual $r(t)$ simultaneously updates the three state-estimates $\hat{x}_p(t), \hat{x}_d(t), \hat{x}_s(t)$.

The Steady-State KBF for Colored Noise

- Assume that all matrices in the state dynamics and sensor equation are constant, i.e.

(65) $$\frac{dx_p(t)}{dt} = A_p x_p(t) + B_p u(t) + L_p d(t)$$

(66) $$z(t) = C_p x_p(t) + M_p n(t)$$

- We must assume that the plant disturbance $d(t)$ is stationary and is generated by the stable LTI dynamic system ($\xi_d(t)$ stationary WN)

(67) $$\frac{dx_d(t)}{dt} = A_d x_d(t) + B_d \xi_d(t)$$

(68) $$d(t) = C_d x_d(t) + D_d \xi_d(t)$$

- We must also assume that the sensor noise $n(t)$ is stationary and is generated by the stable LTI dynamic system ($\theta_s(t)$ stationary WN)

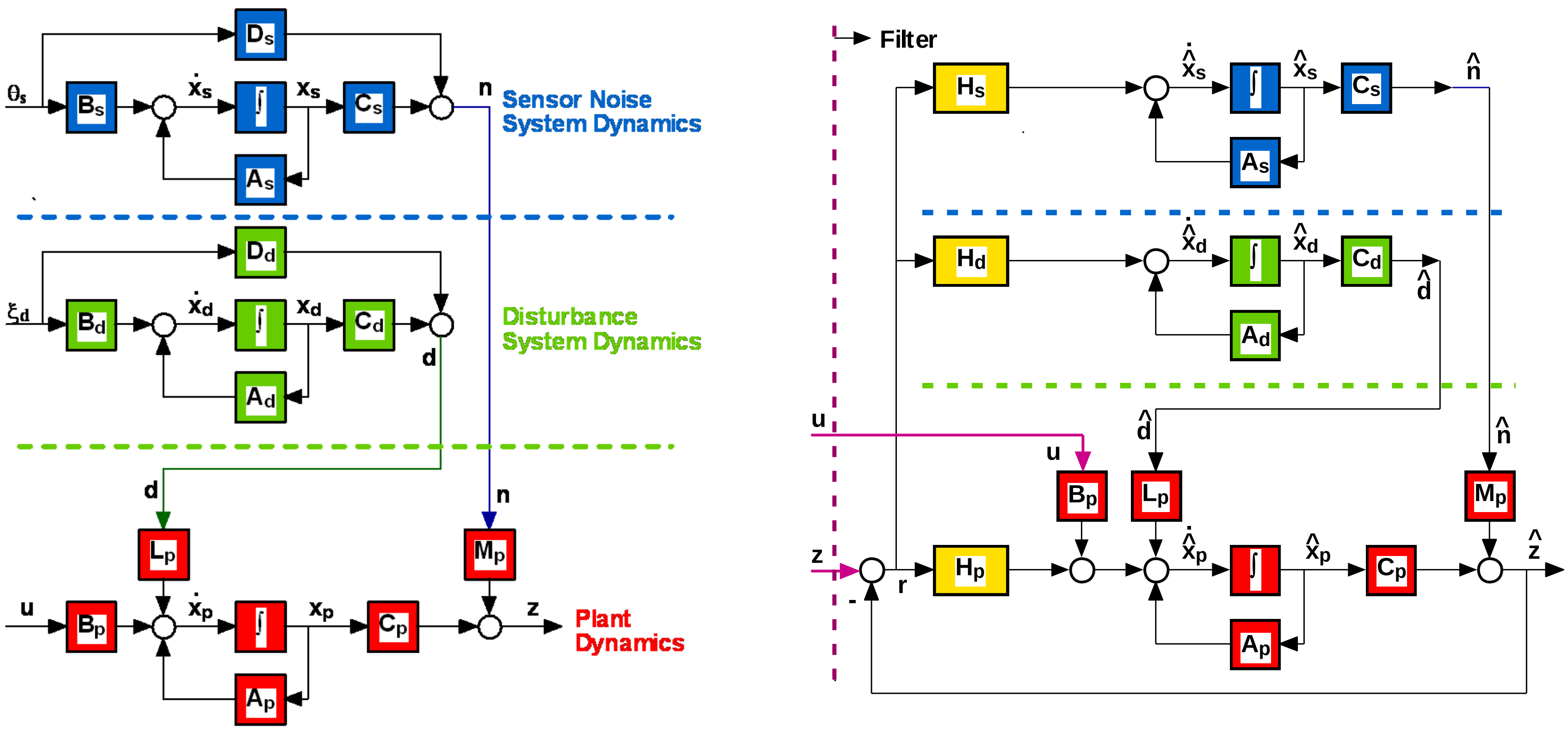
(69) $$\frac{dx_s(t)}{dt} = A_s x_s(t) + B_s \theta_s(t)$$

(70) $$n(t) = C_s x_s(t) + D_s \theta_s(t)$$

Additional Assumptions

- The assumed stability of the dynamic systems that generate the disturbance and the sensor noise satisfy the required stabilizability and detectability assumptions for a well-posed steady-state KBF
- Since the plant may be unstable, we need to be sure that all unstable modes of the plant are observable, so the pair $[A_p, C_p]$ must be detectable (the modes of the disturbance dynamics are assumed to be all stable so that they are automatically detectable)
- We also must ensure that all plant unstable modes are controllable from the plant white noise. In this case the plant white noise comes from $\xi_d(t)$ in the disturbance dynamics, so the pair $[A_p, L_p]$ must be stabilizable
- Both matrices D_s and M_p must be invertible, so that there always exists some white noise corrupting the measurements

Visualization



Steady-State KBF Equations

- The plant-state estimates $\hat{x}_p(t)$ are generated by

(71) $$\frac{d\hat{x}_p(t)}{dt} = A_p\hat{x}_p(t) + B_p u(t) + L_p \underbrace{C_d\hat{x}_d(t)}_{\hat{d}(t)} + H_p r(t)$$

- The disturbance-state estimates $\hat{x}_d(t)$ are generated by

(72) $$\frac{d\hat{x}_d(t)}{dt} = A_d\hat{x}_d(t) + H_d r(t); \quad \hat{d}(t) = C_d\hat{x}_d(t)$$

- The sensor noise state estimates $\hat{x}_s(t)$ are generated by

(73) $$\frac{d\hat{x}_s(t)}{dt} = A_s\hat{x}_s(t) + H_s r(t); \quad \hat{n}(t) = C_s\hat{x}_s(t)$$

- The residual vector $r(t)$ is defined by

(74) $$r(t) = z(t) - C_p\hat{x}_p(t) - M_p \underbrace{C_s\hat{x}_s(t)}_{\hat{n}(t)}$$

- The constant KBF gain-matrices are $H_p\left(n_p \times m_p\right)$ $H_d\left(n_d \times m_p\right)$ $H_s\left(n_s \times m_p\right)$ in eqs. (71), (72), (73), respectively

Steady-State Covariances

- For the constant matrix case we use the obvious definitions

(75) $$A \equiv \begin{bmatrix} A_p & L_p C_d & 0 \\ 0 & A_d & 0 \\ 0 & 0 & A_s \end{bmatrix}; \;\; C \equiv \begin{bmatrix} C_p & 0 & M_p C_s \end{bmatrix}; \; \xi(t) \equiv \begin{bmatrix} \xi_d(t) \\ \theta_s(t) \end{bmatrix};$$

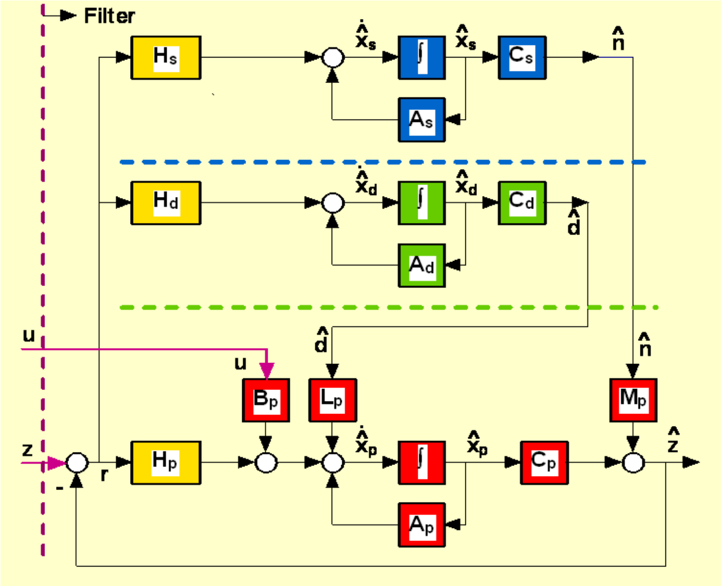
$$\theta(t) \equiv M_p D_s \theta_s(t); \;\; \Xi \equiv \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; \; \Psi \equiv \begin{bmatrix} 0 \\ D_s' M_p' \end{bmatrix}; \; \Theta \equiv M_p D_s D_s' M_p'$$

- We use the above augmented matrices to solve the modified algebraic Riccati equation (34), where Σ_e decomposes to

(76) $$\Sigma_e = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pd} & \Sigma_{ps} \\ \Sigma_{pd}' & \Sigma_{dd} & \Sigma_{ds} \\ \Sigma_{ps}' & \Sigma_{ds}' & \Sigma_{ss} \end{bmatrix}$$

- Note that the diagonal-blocks matrices $\Sigma_{pp}, \Sigma_{dd}, \Sigma_{ss},$ are the covariance matrices of the plant-state, disturbance-state, and sensor-noise-state, respectively

Steady-State KBF Gains



- We use eq. (33) and the augmented matrices to calculate the constant filter-gain matrix H_e

$$(77) \qquad H_e = \begin{bmatrix} H_p \\ H_d \\ H_s \end{bmatrix} \quad \begin{matrix} n_p \times m_p \text{ matrix} \\ n_d \times m_p \text{ matrix} \\ n_s \times m_p \text{ matrix} \end{matrix}$$

which are required to implement the state-estimators of eqs. (71) to (73)

Concluding Remarks

- Accurate modeling of colored disturbance and sensor-noise random processes greatly improves the performance of the KBF
- It turns out that disturbance and/or sensor-noise modeling is particularly useful when one designs optimal feedback control systems that must have superior performance in disturbance-rejection and insensitivity to sensor noise
- Even though the complexity of the KBF increases, due to the augmented dynamics, this is typically justified because of performance gains

References

[1]. A. Gelb, *Applied Optimal Estimation*, MIT Press, 1974, pp. 133-136

[2]. M.S. Grewal and A.P. Andrews, *Kalman Filtering: Theory and Practice*, Prentice-Hall, 1993, pp. 79-83 and 119-120

[3]. B.D.O. Anderson and J.B. Moore, *Optimal Filtering*, Prentice-Hall, 1979, Chapter 11