The Extended Kalman Filter (EKF): The Continuous-Time Case

MICHAEL ATHANS

MIT and ISR/IST

Last Revised: November 9, 2001

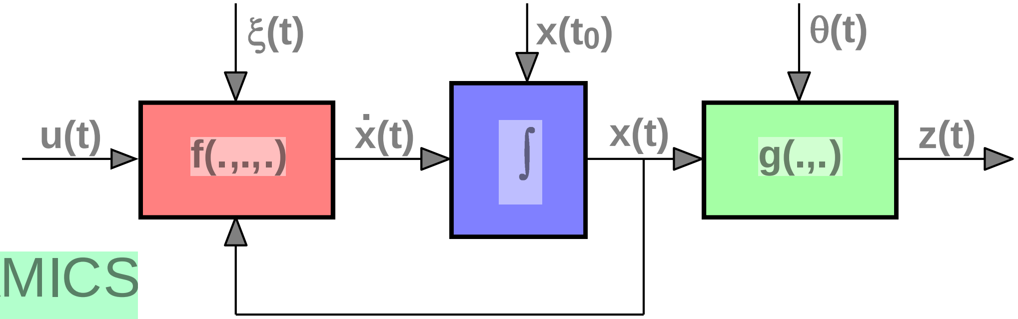
Ref. No. KF #13A

Theme

- First, summarize the results of the extended Kalman filter (EKF) algorithm for the case of continuous-time nonlinear dynamics and continuous-time nonlinear sensor measurements
  - as in the corresponding discrete-time nonlinear EKF algorithm, we must calculate on-line the (pseudo) covariance matrix, using the current value of the estimated state-vector, and then calculate the EKF gain matrix (also on-line)
- Second, summarize the results of the extended Kalman filter (EKF) algorithm for the case of continuous-time nonlinear dynamics and discrete-time nonlinear sensor measurements
  - equations look “messy”, due to time indexing
  - equations split into “predict” and “update” cycles
  - we must calculate on-line the (pseudo) covariance matrix, using the current value of the estimated state-vector, and then calculate the EKF gain matrix (also on-line)

Part I: Continuous Time Dynamics and Continuous Time Measurements

Plant and Sensor Modeling



PLANT - SENSOR DYNAMICS

$$\frac{dx(t)}{dt} = f\big(x(t), u(t), \xi(t)\big)$$

$$z(t) = g\big(x(t), \theta(t)\big)$$

UNCERTAINTY MODELS

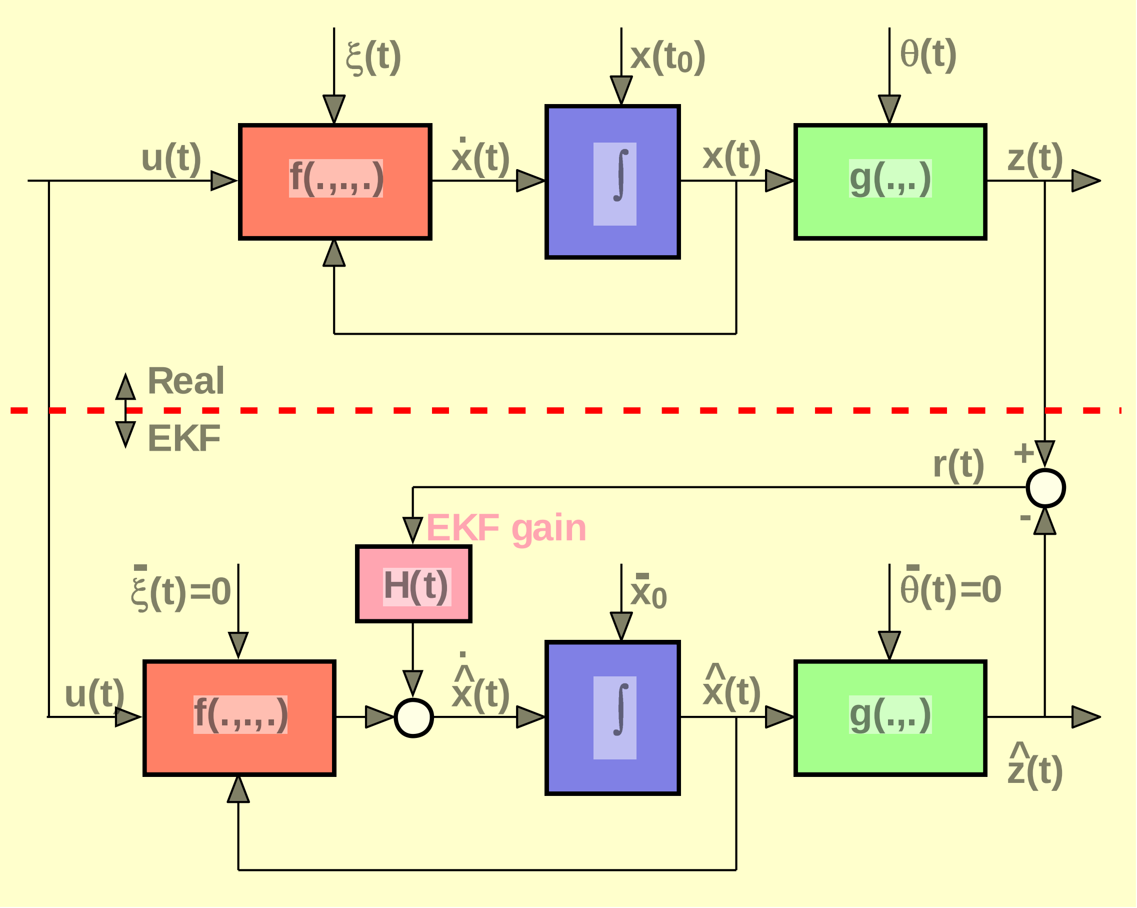
$$E\{x(t_0)\} = \bar{x}_0 \,;\, cov\big[x(t_0); x(t_0)\big] = \Sigma_0$$

$$E\{\xi(t)\} = 0; cov\big[\xi(t); \xi(\tau)\big] = \Xi(t)\delta(t-\tau)$$

$$E\{\theta(t)\} = 0; cov\big[\theta(t); \theta(\tau)\big] = \Theta(t)\delta(t-\tau)$$

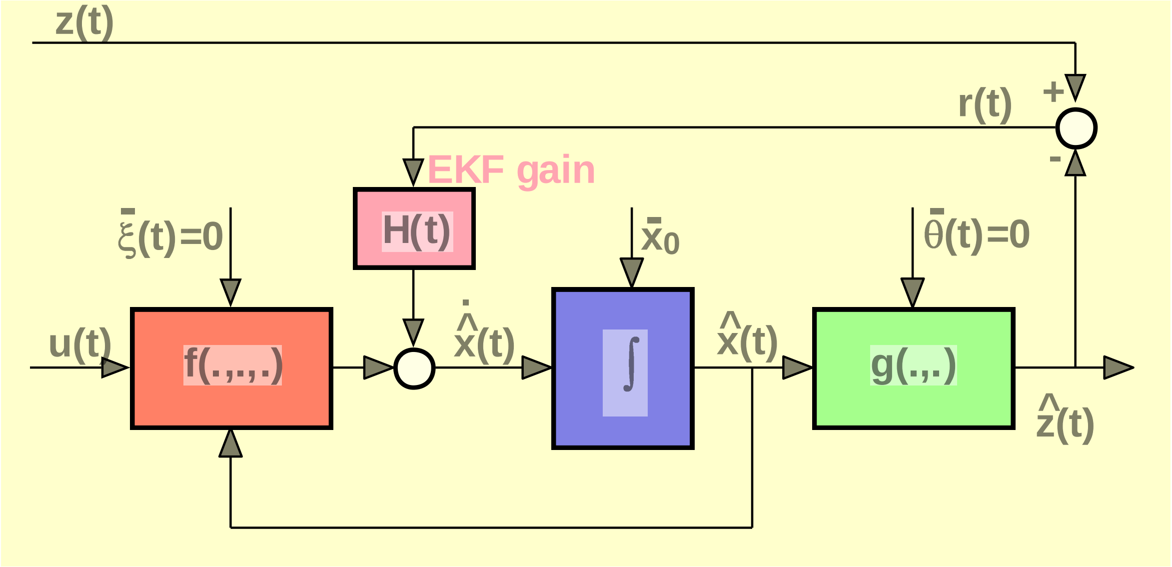
$x(t_0), \xi(t), \theta(\tau)$ independent

Structure of the EKF



- The EKF uses a replica of the nonlinear plant and sensor dynamics, correcting the state-estimate by a linear gain matrix, *H(t),* multiplying the residual vector, *r(t)*

EKF State-Estimate Equations



- State-estimate vector $\hat{x}(t)$ is generated by solving the nonlinear differential equations (on-line)

$$(1) \quad \frac{d\hat{x}(t)}{dt} = f\big(\hat{x}(t), u(t), 0\big) + H(t)\underbrace{\big[z(t) - g\big(\hat{x}(t), 0\big)\big]}_{r(t)}; \quad \hat{x}(t_0) = \bar{x}_0$$

- The EKF gain matrix $H(t)$ must be determined. It turns out that $H(t)$ must be computed on-line

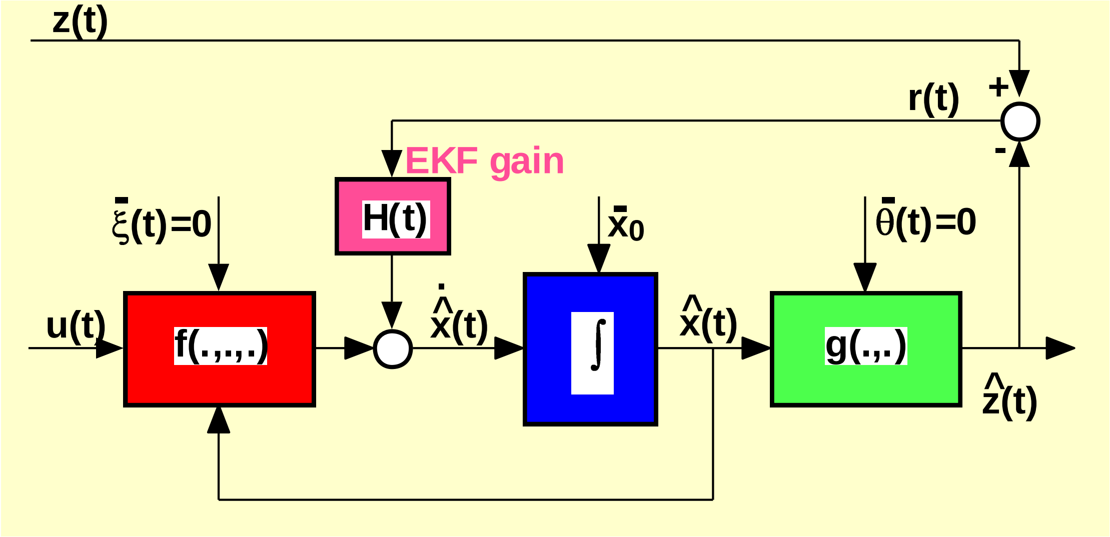
Notation: Jacobian Matrices

- Given the state-nonlinearity, $f\big(x(t), u(t), \xi(t)\big)$, and the sensor nonlinearity, $g\big(x(t), \theta(t)\big)$, we compute on-line the following Jacobian matrices, i.e. matrices of first partial derivatives, using the current EKF state-estimate $\hat{x}(t)$, and the means of $\xi(t), \theta(t)$ which are zero

(1a) $$\hat{A}(t) \equiv \left.\frac{\partial f\big(x, u, \xi\big)}{\partial x}\right|_{x=\hat{x}(t), u(t), 0} ; \quad \hat{L}(t) \equiv \left.\frac{\partial f\big(x, u, \xi\big)}{\partial \xi}\right|_{x=\hat{x}(t), u(t), 0}$$

$$\hat{C}(t) \equiv \left.\frac{\partial g\big(x, \theta\big)}{\partial x}\right|_{x=\hat{x}(t), 0} ; \quad \hat{D}(t) \equiv \left.\frac{\partial g\big(x, \theta\big)}{\partial \theta}\right|_{x=\hat{x}(t), 0}$$

EKF Gain Matrix Calculation



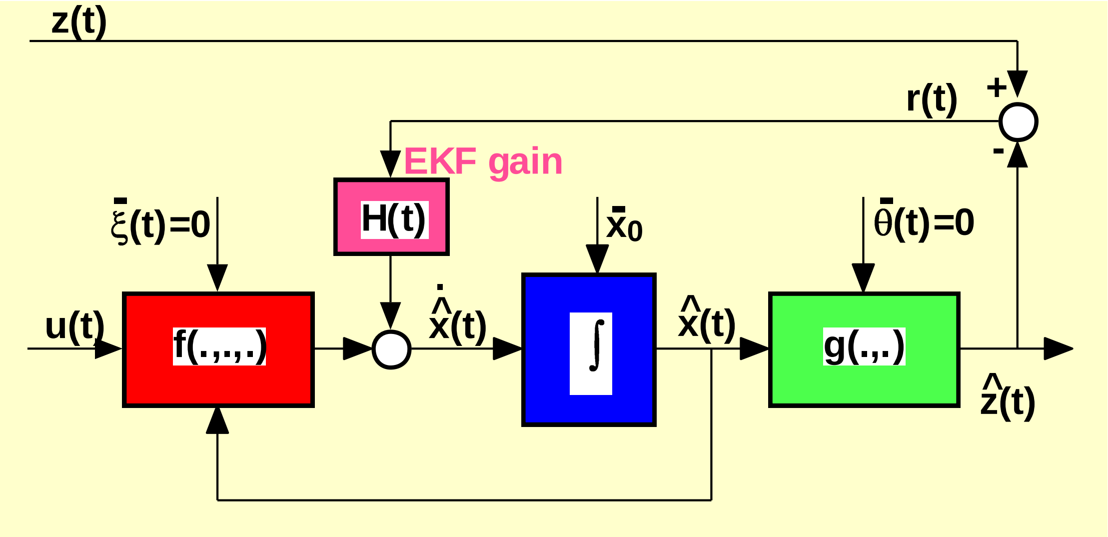
- First, solve on-line the (pseudo) covariance matrix differential equation

(2) $$\frac{d\Sigma(t)}{dt} = \hat{A}(t)\Sigma(t) + \Sigma(t)\hat{A}'(t) + \hat{L}(t)\Xi(t)\hat{L}'(t)$$
$$- \Sigma(t)\hat{C}'(t)\left[\hat{D}(t)\Theta(t)\hat{D}'(t)\right]^{-1}\hat{C}(t)\Sigma(t); \quad \Sigma(t_0) = \Sigma_0$$

- Then calculate on-line the EKF gain matrix $H(t)$

(3) $$H(t) = \Sigma(t)\hat{C}'(t)\left[\hat{D}(t)\Theta(t)\hat{D}'(t)\right]^{-1}$$

EKF Residual



- The EKF residual is (approximately) zero-mean continuous-time white noise

$$r(t) \equiv z(t) - g(\hat{x}(t), 0)$$

$$E\{r(t)\} \cong 0$$

$$cov[r(t); r(\tau)] \cong E\{r(t)r'(\tau)\} = \hat{D}(t)\Theta(t)\hat{D}'(t)\delta(t-\tau)$$

Concluding Remarks

- The extended Kalman filter (EKF) is a suboptimal estimator
  - the optimal state estimator cannot be implemented, because it requires on-line solution of nonlinear partial differential equations, for the nongaussian conditional pdf of the state
- Unlike the linear Kalman-Bucy filter, the EKF requires the on-line calculation of the solution matrix, $\Sigma(t)$, of the (pseudo) covariance matrix differential equation so as to calculate, also on-line, the required EKF gain matrix, $H(t)$
- It is possible to also derive the 2nd order filter equations
- Thus, for an n-dimensional state equation, we need to solve a total of $n+[n(n+1)/2]$ nonlinear, time-varying, scalar differential equations
  - n to generate the state-estimate vector
  - $n(n+1)/2$ to generate the symmetric positive-semidefinite (pseudo) covariance matrix, $\Sigma(t)$

Part II: Continuous Time Dynamics and Discrete Time Measurements

Continuous-Time Dynamics, Discrete-Time Measurements

- Given previous development, it is easy to establish the EKF equations when
  - the plant is described by continuous-time nonlinear stochastic differential equations
  - the noisy measurements occur only at discrete instants of time and sensors are nonlinear
- The notation gets a bit messy, but the concepts follow straight-forward extensions of previous material
- It is convenient to separate the calculations into an "one-step predict cycle" and an "update cycle", as in the discrete-time linear Kalman filter

Mathematical Modeling

- Continuous - time nonlinear plant

(4) $$\frac{dx(t)}{dt} = f\left(x(t), u(t), \xi(t)\right)$$

- Nonlinear measurements at discrete times: $t_1, t_2, \ldots, t_k, \ldots$

(5) $$z(t_{k+1}) = g\left(x(t_{k+1}), \theta(t_{k+1})\right); \quad k = 0, 1, 2, \ldots$$

- Uncertainty models:

(6) $$E\left\{x(t_0)\right\} = \bar{x}_0; \quad cov\left[x(t_0); x(t_0)\right] = \Sigma_0$$

(7) $$E\left\{\xi(t)\right\} = 0; \quad cov\left[\xi(t); \xi(\tau)\right] = \Xi(t)\delta(t-\tau)$$

(8) $$E\left\{\theta(t_k)\right\} = 0; \quad cov\left[\theta(t_k); \theta(t_j)\right] = \Theta(t_k)\delta_{t_k t_j}$$

(9) $x(t_0),\ \xi(t),\ \theta(t_k)$ independent

One-Step Predict Cycle

- State - estimate at $t = t_{k+1}, \quad \hat{x}(t_{k+1} \mid t_k),$ before measurement $z(t_{k+1}),$ is solution at $t = t_{k+1}$ of the nonlinear differential equation

(10) $$\frac{d\hat{x}(t)}{dt} = f(\hat{x}(t), u(t), 0); \quad \hat{x}(t_k) = \hat{x}(t_k \mid t_k); \quad \hat{x}(t_0) = \hat{x}(t_0 \mid t_0) = \bar{x}_0$$

- Covariance matrix at $t = t_{k+1}, \quad \Sigma(t_{k+1} \mid t_k),$ before measurement $z(t_{k+1}),$ is solution at $t = t_{k+1}$ of the nonlinear matrix differential equation

(11) $$\frac{d\Sigma(t)}{dt} = \hat{A}(t)\Sigma(t) + \Sigma(t)\hat{A}'(t) + \hat{L}(t)\Xi(t)\hat{L}'(t),$$

$$\Sigma(t_k) = \Sigma(t_k \mid t_k); \quad \Sigma(t_0) = \Sigma(t_0 \mid t_0) = \Sigma_0$$

- In eq. (11) the Jacobian matrices $\hat{A}(t), \hat{L}(t),$ are as defined before

Update Cycle

- After the measurement $z(t_{k+1})$ is obtained, the state estimate and covariance are updated from their one-step predicted values
- Updated covariance matrix: $\Sigma(t_{k+1} \mid t_{k+1})$

(12) $\Sigma(t_{k+1} \mid t_{k+1}) = \Sigma(t_{k+1} \mid t_k) - \Sigma(t_{k+1} \mid t_k)\hat{C}'(t_{k+1}) \cdot$

$\cdot\left[\hat{C}(t_{k+1})\Sigma(t_{k+1} \mid t_k)\hat{C}'(t_{k+1}) + \hat{D}(t_{k+1})\Theta(t_{k+1})\hat{D}'(t_{k+1})\right]^{-1} .\hat{C}(t_{k+1})\Sigma(t_{k+1} \mid t_k)$

where the Jacobian matrices, $\hat{C}(t_{k+1}), \hat{D}(t_{k+1}),$ are as defined before and evaluated at the predicted estimate, $x(t_{k+1}) = \hat{x}(t_{k+1} \mid t_k)$

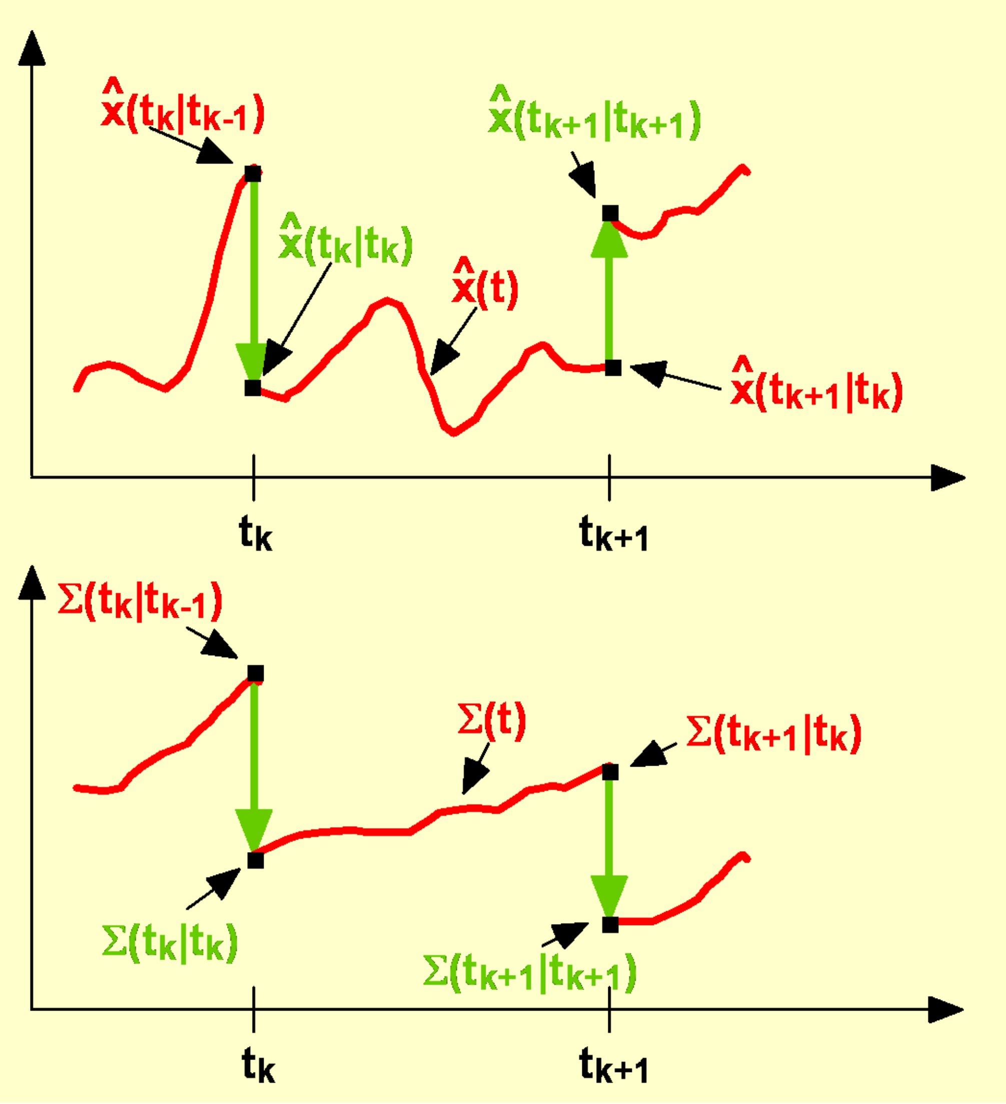
- The EKF gain matrix, $H(t_{k+1}),$ is calculated by

(13) $H(t_{k+1}) = \Sigma(t_{k+1} \mid t_{k+1})\hat{C}'(t_{k+1}) \cdot \left[\hat{D}(t_{k+1})\Theta(t_{k+1})\hat{D}'(t_{k+1})\right]^{-1}$

- The updated state estimate $\hat{x}(t_{k+1} \mid t_{k+1})$ is given by

(14) $\hat{x}(t_{k+1} \mid t_{k+1}) = \hat{x}(t_{k+1} \mid t_k) + H(t_{k+1})\left[z(t_{k+1}) - g\left(\hat{x}(t_{k+1} \mid t_k), 0\right)\right]$

Visualization



Concluding Remarks

- Most physical applications require the implementation of EKFs with continuous-time nonlinear plant dynamics and (linear or nonlinear) discrete-time measurements
  - example: radar tracking of orbiting satellites
  - example: radar or ladar tracking of aircraft or missiles
  - example: sonar tracking of ships or submarines
  - example: viral-infection population estimates using infrequent blood tests
  - example: ecological system sampling of competing species
- So, results presented are very useful in practice
- Once more, it is possible to derive the 2nd-order filter equations

References

[1]. A. Gelb, *Applied Optimal Estimation*, MIT Press, 1974, Chapter 6

[2]. M.S. Grewal and A.P. Andrews, *Kalman Filtering*, Prentice-Hall, 1993, Chapter 5

[3]. P.S. Maybeck, *Stochastic Models, Estimation, and Control,* Vols. I and II, Academic Press, 1979 and 1982

[4]. A.H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, 1970

[5]. H.W. Sorenson (ed), *Kalman Filtering: Theory and Applications,* IEEE Press, 1985