Suboptimal Nonlinear Filtering: The Discrete-Time Case

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Last Revised: November 18, 2001

Ref. No. KF #13

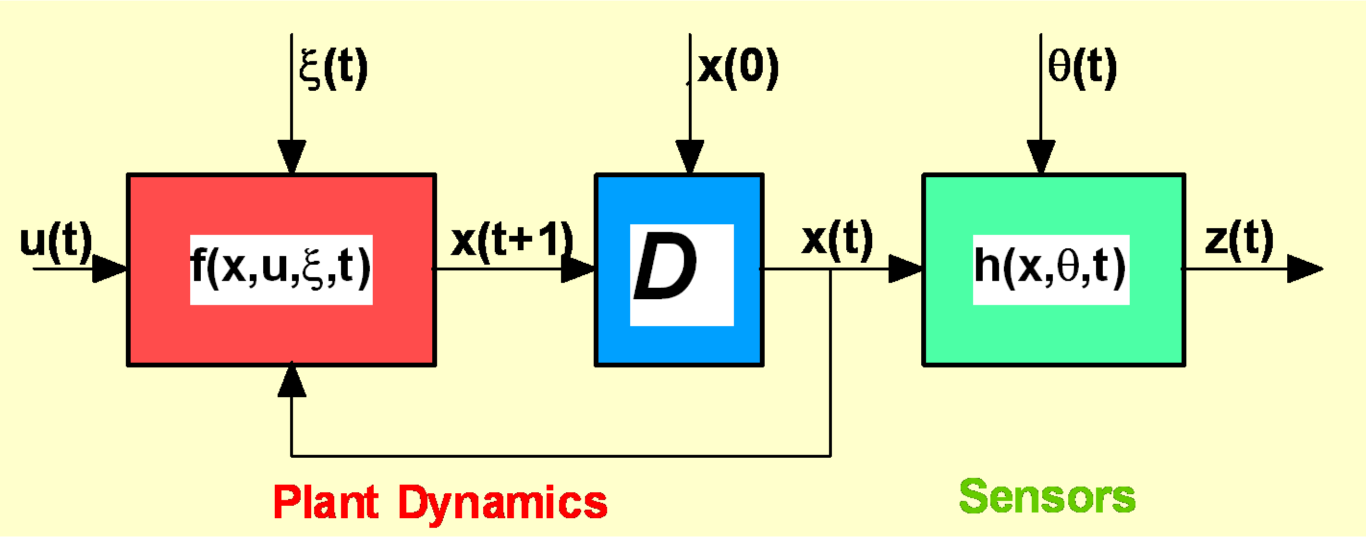
The Bad News

- If the state dynamics and/or the sensor equations are nonlinear, it is computationally infeasible to compute an “optimal” state-estimate, i.e. to calculate in real-time the true conditional mean of the state and its associated conditional covariance matrix
- The basic reason is that even if we make gaussian assumptions on the plant state, the plant white noise sequence, and the sensor white noise sequence, the conditional probability density function $p(x(t)|Z(t))$ is not gaussian
  - therefore, its dynamic evolution cannot be described in terms of its mean and covariance
- Therefore, for nonlinear estimation and filtering problems we are forced to use suboptimal algorithms
- The sophistication of the nonlinear filtering algorithm used will depend on the amount of real-time computational resources available

Theme

- We concentrate on general discrete-time nonlinear dynamical systems and nonlinear noisy sensors
  - we shall examine later systems with continuous-time dynamics
- We present, discuss and summarize the two most popular suboptimal nonlinear filtering algorithms, [1]-[6]
  - the "Extended Kalman Filter (EKF)"
  - the "Second-Order Filter (SOF)", or Gaussian filter
- The SOF requires a modest increase in the real-time computational requirements as compared with the EKF
- Unlike the linear Kalman filter case, both the EKF and the SOF require the on-line calculation of both the (pseudo) covariance matrix equations and of the nonlinear filter gain-matrix
- There is no a-priori guarantee that either the EKF or the SOF will "work well" in a particular application

Plant and Sensor Model

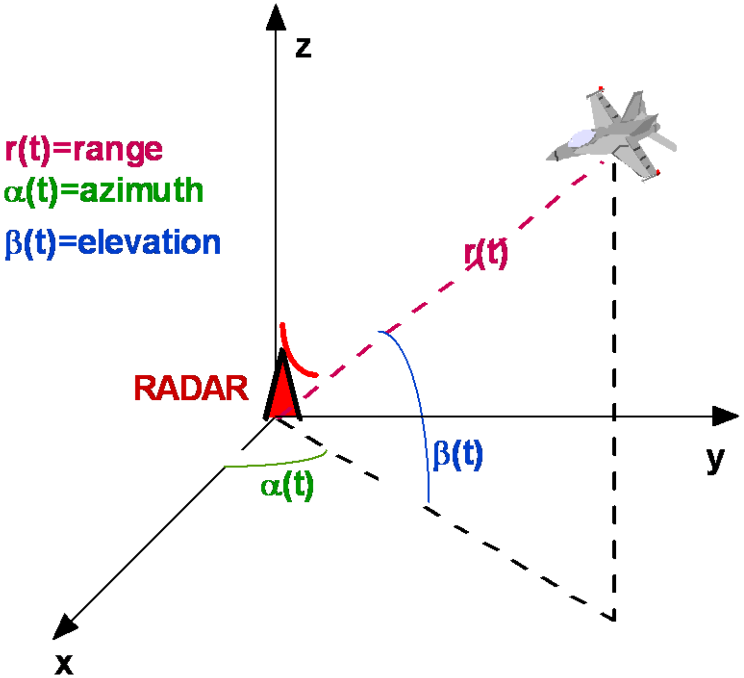


- Discrete-time index: $t = 0, 1, 2, \ldots$
- Nonlinear state-dynamics: $x(t+1) = f\left(x(t), u(t), \xi(t), t\right)$
- Nonlinear sensor measurements: $z(t+1) = h\left(x(t+1), \theta(t+1), t+1\right)$

Aircraft Tracking

PROBLEM

- Estimate aircraft positions and velocities in inertial coordinates based on noisy radar measurements
- Aircraft dynamics are nonlinear (exponential atmosphere, quadratic - in - speed drag forces)



RADAR MEASUREMENTS

Range: $R = \sqrt{x^2 + y^2 + z^2}$

Azimuth: $\alpha = tan^{-1} \frac{y}{x}$

Elevation: $\beta = tan^{-1} \frac{z}{\sqrt{x^2 + y^2}}$

- Measurement noise intensity may depend on the state (state - dependent noise)
- Typically noise variance increases with range

Submarine Tracking

- Passive sonar tracking: measure only azimuth

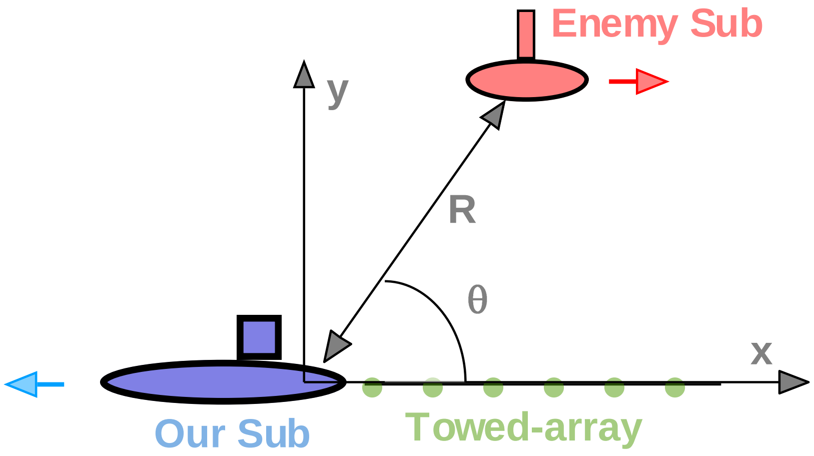
$$\theta = tan^{-1}\frac{y}{x}$$

- Active sonar tracking: measure range

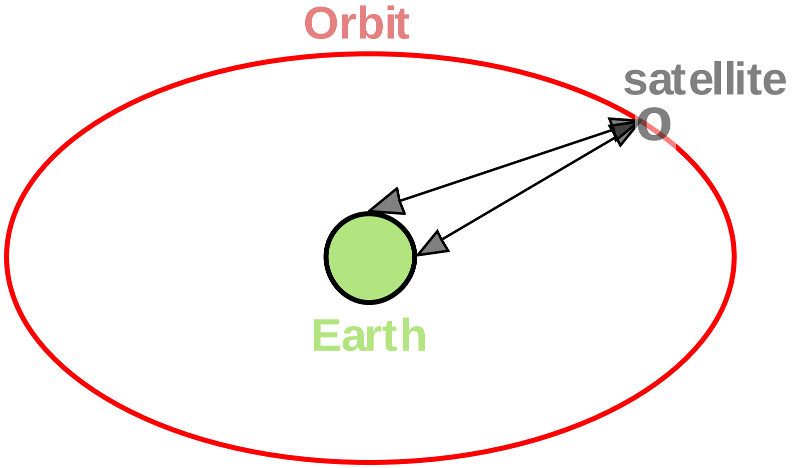
$$R = \sqrt{x^2 + y^2}$$

and azimuth

- Variance of sensor noise may depend on range



Satellite Tracking



- Satellite dynamics are nonlinear, due to inverse-square law
- One or more radars measure the range to the satellite, and associated azimuth and elevations angles
- All measurements are nonlinear functions of the satellite inertial coordinates

On-Line Parameter Estimation

- Given a linear discrete-time system with one or more uncertain parameters
- If we are interested in estimating the parameters, in addition to the state variables, we have a nonlinear estimation problem
- Example: a is a scalar uncertain parameter in 1st-order LTI system

$$x(t+1) = ax(t) + \xi(t); \quad z(t+1) = x(t+1) + \theta(t+1)$$

define: $x_1(t) \equiv x(t); \quad x_2(t) \equiv a.$ It follows that:

$$\left.\begin{aligned} x_1(t+1) &= x_1(t) \cdot x_2(t) + \xi(t) \\ x_2(t+1) &= x_2(t) \end{aligned}\right\}$$

$$z(t+1) = x_1(t+1) + \theta(t+1)$$

which represents a nonlinear filtering problem

Historical Perspective

- P. Swerling in 1959 published the first paper that uses an EKF algorithm for satellite orbit estimation, Ref. [5]
  - Swerling's paper preceded the publication of the Kalman filter papers by more than a year
- R.E. Kalman had nothing to do with the EKF nonlinear filter
- The navigation system of the manned Apollo mission to the moon was based upon the EKF algorithm
- Space navigation, satellite orbit determination, inertial navigation systems and surveillance systems for aircraft, ships, submarines and missiles provided a very fertile ground for the explosive development, during the 1960's, in the theory, algorithms and applications of nonlinear filtering

Mathematical Modeling

- Time-index: $t = 0, 1, 2, ..$
- State dynamics

(1) $x(t+1) = f\big(x(t), u(t), \xi(t), t\big)$

$$x(t) \in R^n, u(t) \in R^q, \xi(t) \in R^p$$

- Measurements

(2) $z(t+1) = h(x(t+1), \theta(t+1), t+1)$

$$z(t) \in R^m, \theta(t) \in R^m$$

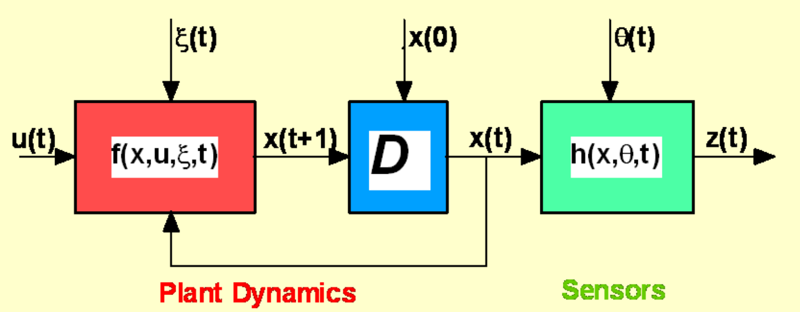
- Initial state $x(0) \sim N\big(\bar{x}_0, \Sigma_0\big)$
- Plant white noise is gaussian

(3) $E\{\xi(t)\} = 0,\ E\{\xi(t)\xi'(\tau)\} = \Xi(t)\delta_{t\tau}$

- Sensor white noise is gaussian

(4) $E\{\theta(t)\} = 0,\ E\{\theta(t)\theta'(\tau)\} = \Theta(t)\delta_{t\tau}$

- $x(0), \xi(t), \theta(\tau)$ independent $\forall t, \tau$



FURTHER ASSUMPTIONS

- The functions $f\big(x, u, \xi, t\big)$ and $h\big(x, \theta, t\big)$ are continuous and continuously differentiable
- The inverse function

$$\theta = h^{-1}(x, z, t)$$

must exist for all x, t

The Conditional Density Function

- Let the set of past controls and measurements be denoted by $Z(t)$

(5) $Z(t) \equiv \{z(1), z(2), \ldots, z(t); u(0), u(1), \ldots, u(t-1)\}$

- Ideally (as in the linear case) we would like to calculate the true conditional pdf of the state $p(x(t) \mid Z(t))$ so that we can use as state-estimate the conditional mean, $E\{x(t) \mid Z(t)\},$ and compute the conditional covariance, $cov[x(t); x(t) \mid Z(t)]$
- Even though the initial state, plant noise and sensor noise were assumed gaussian, the state nonlinearity $f(.,.,.,.)$ and / or the sensor nonlinearity $h(.,.,.)$ "destroy" the gaussian characteristics
- In general, $p(x(t) \mid Z(t))$ is not gaussian, and its dynamic evolution requires the on-line solution of complex partial differential equations
- Therefore, we cannot compute on-line the desired conditional mean $E\{x(t) \mid Z(t)\}$ and conditional covariance $cov[x(t); x(t) \mid Z(t)]$
- We must be satisfied with sub-optimal algorithms to obtain a state-estimate, denoted by $\hat{x}(t \mid t),$ and associated covariance $\Sigma(t \mid t)$

The Basic Problem

- Suppose that x is a scalar-valued random variable with pdf $p(x)$
- Then the expected value (mean) is given by

(6) $$\bar{x} \equiv E\{x\} \equiv \int_{-\infty}^{\infty} xp(x)dx$$

- Let $f(x)$ be a nonlinear function of the random variable x. Then

(7) $$E\{f(x)\} \equiv \int_{-\infty}^{\infty} f(x)p(x)dx$$

- However,

(8) $$E\{f(x)\} \neq f\left(E\{x\}\right) = f(\bar{x})$$

Taylor Series Approximations

- Assuming that the function $f(x)$ is smooth and continuously differentiable, we can use a Taylor series expansion of $f(x)$ about $x = \bar{x}$, i.e.

(9) $$f(x) = f(\bar{x}) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=\bar{x}} \cdot (x - \bar{x}) + \frac{1}{2!} \cdot \left.\frac{\partial f^2(x)}{\partial x^2}\right|_{x=\bar{x}} \cdot (x - \bar{x})^2 + h.o.t \Rightarrow$$

(10) $$E\{f(x)\} \approx f(\bar{x}) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=\bar{x}} \cdot E\{\ (x - \bar{x})\ \} + \frac{1}{2!} \cdot \left.\frac{\partial f^2(x)}{\partial x^2}\right|_{x=\bar{x}} \cdot E\{\ (x - \bar{x})^2\ \}$$

- But $E\{(x - \bar{x})\} = 0,$ and letting Σ denote the variance, $\Sigma = E\{(x - \bar{x})^2\}$

(11) $$E\{f(x)\} \approx f(\bar{x}) + \frac{1}{2!} \cdot \left.\frac{\partial f^2(x)}{\partial x^2}\right|_{x=\bar{x}} \cdot \Sigma$$

- We note that the approximation is valid if indeed the third and higher - order terms in the Taylor series expansion can be neglected. Indeed, if we also neglect the quadratic terms we obtain the simple relation

(12) $$E\{f(x)\} \approx f(\bar{x})$$

Discussion

- In the sequel, we shall discuss two different (but related) nonlinear filtering algorithms
  - the simplest is the "Extended Kalman Filter" or EKF
  - the more complicated is the "Second-Order Filter" or SOF
- Because we deal with vector-valued random variables the notation will get more complicated. However, roughly speaking, the EKF will use the simpler approximation given by eq. (12), i.e.

(13) $$E\{f(x)\} \cong f(\bar{x})$$

while the SOF will use the approximation given by eq. (11), i.e.

(14) $$E\{f(x)\} \cong f(\bar{x}) + \frac{1}{2} \cdot \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=\bar{x}} \cdot \Sigma$$

where $\bar{x}$ is the mean and Σ is the variance of x, i.e.

(16) $$\bar{x} \equiv E\{x\}, \quad \Sigma \equiv E\left\{(x-\bar{x})^2\right\}$$

Notation: State-Estimates and Covariances

- For both the EKF and the SOF we use the notation (as in linear case)

  $\hat{x}(t \mid t) =$ updated estimate of $x(t)$ given data set $Z(t)$ - see eq. (5)

  $\hat{x}(t+1 \mid t) =$ predicted estimate of $x(t+1)$ given data set $\{u(t), Z(t)\}$,

  i.e. before the measurement $z(t+1)$ is obtained
- We hope that these approximate the true conditional means, i.e.

(17) $\hat{x}(t \mid t) \cong E\{x(t) \mid Z(t)\}$

(18) $\hat{x}(t+1 \mid t) \cong E\{x(t+1) \mid u(t), Z(t)\}$

- Similar notation is used for the state (pseudo) covariances

  $\Sigma(t \mid t) =$ updated covariance of $x(t)$ given data set $Z(t)$

  $\Sigma(t+1 \mid t) =$ predicted covariance of $x(t+1)$ given data set $\{u(t), Z(t)\}$

  and we hope that

(19) $\Sigma(t \mid t) \cong cov[x(t); x(t) \mid Z(t)]$

(20) $\Sigma(t+1 \mid t) \cong cov[x(t+1); x(t+1) \mid u(t), Z(t)]$

Predict Cycle Comparisons

- REAL SYSTEM

(21) $x(t+1) = f\big(x(t), u(t), \xi(t), t\big)$

- EXTENDED KALMAN FILTER

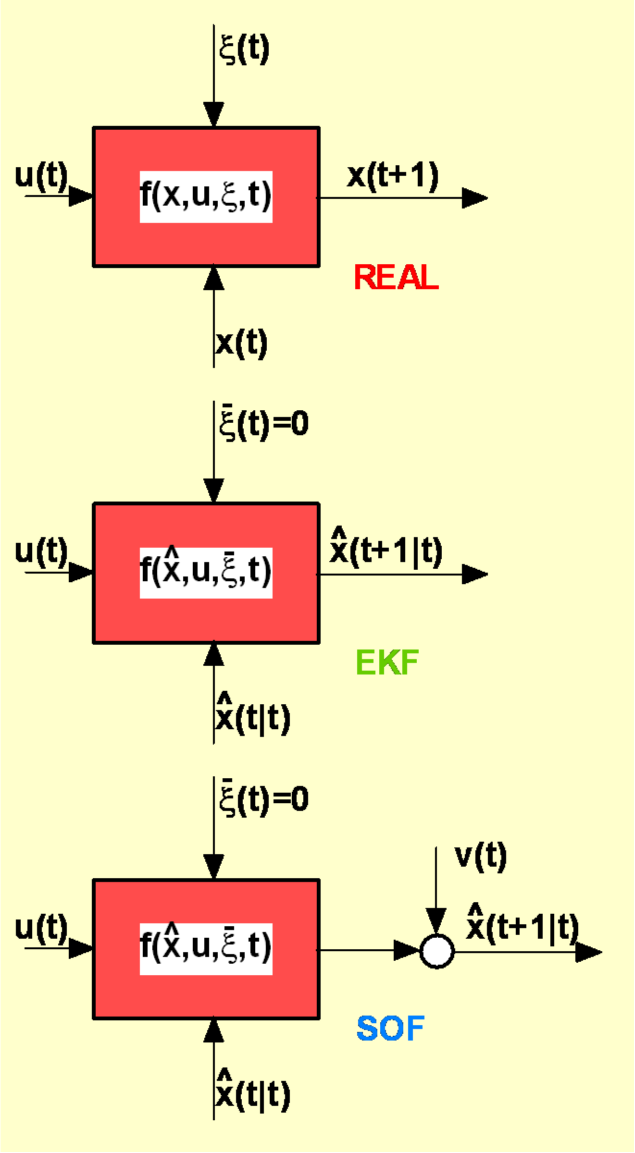
(22) $\hat{x}(t+1 \mid t) = f\big(\hat{x}(t \mid t), u(t), 0, t\big)$

- SECOND-ORDER FILTER

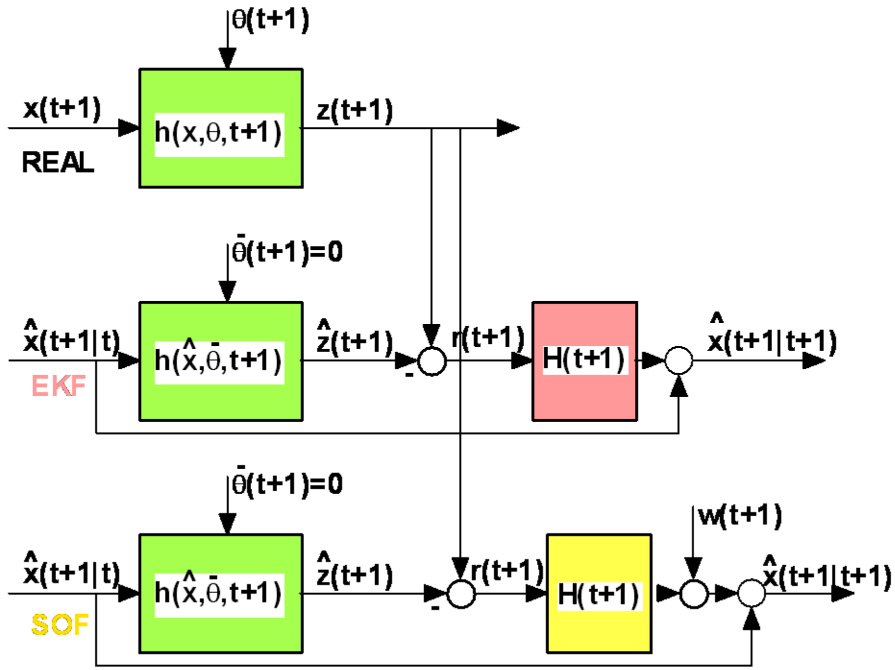
(23) $\hat{x}(t+1 \mid t) = f\big(\hat{x}(t \mid t), u(t), 0, t\big) + v(t)$

$v(t)$ = predict-bias (to be found)

- Both the EKF and the SOF use the nonlinear state dynamics to generate the predict estimate from the updated estimate



Update Cycle Comparisons



- REAL SYSTEM

(24) $z(t+1) = h(x(t+1), \theta(t+1), t+1)$

- RESIDUAL (both EKF and SOF)

(25) $r(t+1) = z(t+1) - \hat{z}(t+1) =$

$= z(t+1) - h(x(t+1), 0, t+1)$

- EKF update

(26) $\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) +$

$+ H(t+1) r(t+1)$

- SOF update: $w(t+1)$ update - bias

(27) $\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) +$

$+ H(t+1) r(t+1) + w(t+1)$

Vector Taylor Series Expansions, I

- We deal with vector - valued functions of a vector

(28) $x \in R^m; \quad g(x): R^m \rightarrow R^n$

(29) $$g(x) \equiv \begin{bmatrix} g_1(x) \\ g_2(x) \\ \cdots \\ g_n(x) \end{bmatrix} \equiv \begin{bmatrix} g_1(x_1, x_2, \ldots, x_m) \\ g_2(x_1, x_2, \ldots, x_m) \\ \cdots \\ g_n(x_1, x_2, \ldots, x_m) \end{bmatrix}$$

- Jacobian matrix (matrix of first partial derivatives)

(30) $$\frac{\partial g(x)}{\partial x} \equiv \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} & \cdots & \frac{\partial g_1(x)}{\partial x_m} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} & \cdots & \frac{\partial g_2(x)}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_n(x)}{\partial x_1} & \frac{\partial g_n(x)}{\partial x_2} & \cdots & \frac{\partial g_n(x)}{\partial x_m} \end{bmatrix}$$ an $n \times m$ matrix

Vector Taylor Series Expansions, II

- Hessian matrix (matrix of second partial derivatives). Consider the k - th scalar element of the vector $g(x)$, $g_k(x) = g_k(x_1, x_2, \ldots, x_m)$

(31) $$\frac{\partial^2 g_k(x)}{\partial x^2} \equiv \begin{bmatrix} \frac{\partial^2 g_k(x)}{\partial {x_1}^2} & \frac{\partial^2 g_k(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 g_k(x)}{\partial x_1 \partial x_m} \\ \frac{\partial^2 g_k(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 g_k(x)}{\partial {x_2}^2} & \cdots & \frac{\partial^2 g_k(x)}{\partial x_2 \partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 g_k(x)}{\partial x_m \partial x_1} & \frac{\partial^2 g_k(x)}{\partial x_m \partial x_2} & \cdots & \frac{\partial^2 g_k(x)}{\partial {x_m}^2} \end{bmatrix}; \quad k = 1, 2, \ldots, n$$

- The Hessian matrix $\frac{\partial^2 g_k(x)}{\partial x^2}$ is a symmetric $m \times m$ matrix, because

(32) $$\frac{\partial^2 g_k(x)}{\partial x_i \partial x_j} = \frac{\partial^2 g_k(x)}{\partial x_j \partial x_i} \quad \forall i, j = 1, 2, \ldots, m; \quad \forall k = 1, 2, \ldots, n$$

Vector Taylor Series Expansions, III

- Multivariable Taylor series expansion of $g(x)$ about $x = x_0$

(33) $$g(x) = g(x_0) + \left.\frac{\partial g(x)}{\partial x}\right|_{x=x_0} \cdot (x - x_0) +$$

$$+\frac{1}{2}\sum_{k=1}^{n} e_k \cdot (x - x_0)' \cdot \left.\frac{\partial^2 g(x)}{\partial x^2}\right|_{x=x_0} \cdot (x - x_0) + h.o.t.$$

where $h.o.t.$ means "higher - order terms" and $e_k \in R^n$ is a "unit" column vector, with 0 in each element, except in the k - th row, i.e.

(34) $$e_k \equiv \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{bmatrix} \quad k - th\ row$$

Jacobian Matrices for EKF and SOF

- State-dynamics nonlinearity: $f(x,u,\xi,t)$. The Jacobian matrices $\hat{A}(t)$ and $\hat{L}(t)$ are evaluated at the updated state-estimate $\hat{x}(t \mid t)$

(35) $$\hat{A}(t) \equiv \left. \frac{\partial f(x,u,\xi,t)}{\partial x} \right|_{x=\hat{x}(t|t),\,\xi=0} ; \quad \hat{A}(t): n \times n$$

(36) $$\hat{L}(t) \equiv \left. \frac{\partial f(x,u,\xi,t)}{\partial \xi} \right|_{x=\hat{x}(t|t),\,\xi=0} ; \quad \hat{L}(t): n \times p$$

- Sensor nonlinearity: $h(x,\theta,t)$. The Jacobian matrices $\hat{C}(t+1)$ and $\hat{D}(t+1)$ are evaluated at the predicted state-estimate $\hat{x}(t+1 \mid t)$

(37) $$\hat{C}(t+1) \equiv \left. \frac{\partial h(x,\theta,t)}{\partial x} \right|_{x=\hat{x}(t+1|t),\,\theta=0} ; \quad \hat{C}(t+1): m \times n$$

(38) $$\hat{D}(t+1) \equiv \left. \frac{\partial h(x,\theta,t)}{\partial \theta} \right|_{x=\hat{x}(t+1|t),\,\theta=0} ; \quad \hat{D}(t+1): m \times m$$

(39) $\hat{D}^{-1}(t+1)$ exists

Hessian Matrices for the SOF, I

- State nonlinearity: $f(x,u,\xi,t)$ with k - th row $f_k(x,u,\xi,t)$; $k=1,2,\ldots,n$

(40) $$f(x,u,\xi,t)=\begin{bmatrix} f_1(x,u,\xi,t) \\ f_2(x,u,\xi,t) \\ \ldots \\ f_n(x,u,\xi,t) \end{bmatrix}$$

(41) $$\hat{F}_k(t) \equiv \left.\frac{\partial^2 f_k(x,u,\xi,t)}{\partial x^2}\right|_{x=\hat{x}(t|t),u(t),\xi=0} \quad ; \quad n \times n \text{ matrix}$$

(42) $$\hat{G}_k(t) \equiv \left.\frac{\partial^2 f_k(x,u,\xi,t)}{\partial \xi^2}\right|_{x=\hat{x}(t|t),u(t),\xi=0} \quad ; \quad p \times p \text{ matrix}$$

(43) $$\hat{N}_k(t) \equiv \left.\frac{\partial^2 f_k(x,u,\xi,t)}{\partial x \partial \xi}\right|_{x=\hat{x}(t|t),u(t),\xi=0} \quad ; \quad n \times p \text{ matrix}$$

Hessian Matrices for the SOF, II

- Sensor nonlinearity: $h(x,\theta,t)$ with j-th row $h_j(x,\theta,t);\ j=1,2,\dots,m$

(44) $$h(x,\theta,t)=\begin{bmatrix} h_1(x,\theta,t) \\ h_2(x,\theta,t) \\ \dots \\ h_m(x,\theta,t) \end{bmatrix}$$

(45) $$\hat{M}_j(t+1)\equiv\left.\frac{\partial^2 h_j(x,\theta,t)}{\partial x^2}\right|_{x=\hat{x}(t+1|t),\theta=0};\quad n\times n \text{ matrix}$$

(46) $$\hat{Q}_j(t+1)\equiv\left.\frac{\partial^2 h_j(x,\theta,t)}{\partial \theta^2}\right|_{x=\hat{x}(t+1|t),\theta=0};\quad m\times m \text{ matrix}$$

(47) $$\hat{R}_j(t+1)\equiv\left.\frac{\partial^2 h_j(x,\theta,t)}{\partial x\partial \theta}\right|_{x=\hat{x}(t+1|t),\theta=0};\quad n\times m \text{ matrix}$$

The EKF Equations: Summary

PREDICT CYCLE

- State predict estimate:

(48) $\hat{x}(t+1 \mid t) = f\big(\hat{x}(t \mid t), u(t), 0, t\big); \quad \hat{x}(0 \mid 0) = E\{x(0)\} = \bar{x}_0$

- Covariance propagation: $\hat{A}(t), \hat{L}(t)$ evaluated at $\hat{x}(t \mid t)$, eqs. (35),(36)

(49) $\Sigma(t+1 \mid t) = \hat{A}(t)\Sigma(t \mid t)\hat{A}'(t) + \hat{L}(t)\Xi(t)\hat{L}'(t); \ \Sigma(0 \mid 0) = \Sigma_0$

UPDATE CYCLE

- Updated covariance: $\hat{C}(t+1), \hat{D}(t+1)$ evaluated at $\hat{x}(t+1 \mid t)$, eqs. (37),(38)

(50) $\Sigma(t+1 \mid t+1) = \Sigma(t+1 \mid t) - \Sigma(t+1 \mid t)\hat{C}'(t+1) \cdot$

$\cdot\left[\hat{C}(t+1)\Sigma(t+1 \mid t)\hat{C}'(t+1) + \hat{D}(t+1)\Theta(t+1)\hat{D}'(t+1)\right]^{-1} \cdot \hat{C}(t+1)\Sigma(t+1 \mid t)$

- EKF gain matrix:

(51) $H(t+1) = \Sigma(t+1 \mid t+1)\hat{C}'(t+1)\left[\hat{D}(t+1)\Theta(t+1)\hat{D}'(t+1)\right]^{-1}$

- State update estimate:

(52) $\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) + H(t+1)\big[z(t+1) - h\big(\hat{x}(t+1 \mid t), 0, t+1\big)\big]$

The SOF: Predict Cycle Summary

- State predict estimate: $\hat{F}_k(t), \hat{G}_k(t)$ evaluated at $\hat{x}(t \mid t),$ see (41),(42)

(53) $\hat{x}(t+1 \mid t) = f\big(\hat{x}(t \mid t), u(t), 0, t\big) + v(t);\ \hat{x}(0 \mid 0) = \bar{x}_0;\ \Sigma(0 \mid 0) = \Sigma_0$

(54) $$v(t) = \frac{1}{2}\sum_{k=1}^{n} e_k \cdot \Big[tr\big[\hat{F}_k(t)\Sigma(t \mid t)\big] + tr\big[\hat{G}_k(t)\Xi(t)\big]\Big]$$

- Covariance prediction: $\hat{A}(t), \hat{L}(t)$ evaluated at $\hat{x}(t \mid t),$ see (35),(36)

(55) $\Sigma(t+1 \mid t) = \hat{A}(t)\Sigma(t \mid t)\hat{A}'(t) + \hat{L}(t)\Xi(t)\hat{L}'(t);\quad \Sigma(0 \mid 0) = \Sigma_0$

- Note that the predict-bias term $v(t)$ is significant if at $\hat{x}(t \mid t)$
  - (a) the state nonlinearity $f(.)$ has significant x-direction curvature, $\hat{F}_k(t),$ and state covariance $\Sigma(t \mid t)$ is large, and/or
  - (b) the state nonlinearity $f(.)$ has significant ξ-direction curvature, $\hat{G}_k(t),$ and plant-noise covariance Ξ(t) is large

The SOF: Update Cycle Summary

- Covariance update: $\hat{C}(t+1), \hat{D}(t+1)$ evaluated at $\hat{x}(t+1 \mid t),$ see (37),(38)

(56) $$\Sigma(t+1 \mid t+1) = \Sigma(t+1 \mid t) - \Sigma(t+1 \mid t)\hat{C}'(t+1) \cdot$$

$$\cdot\left[\hat{C}(t+1)\Sigma(t+1 \mid t)\hat{C}'(t+1) + \hat{D}(t+1)\Theta(t+1)\hat{D}'(t+1)\right]^{-1}\hat{C}(t+1)\Sigma(t+1 \mid t)$$

- SOF gain matrix:

(57) $$H(t+1) = \Sigma(t+1 \mid t+1)\hat{C}'(t+1)\left[\hat{D}(t+1)\Theta(t+1)\hat{D}'(t+1)\right]^{-1}$$

- Update - bias term: $\hat{M}_j(t+1), \hat{Q}_j(t+1)$ evaluated at $\hat{x}(t+1 \mid t),$ see (45),(46)

  with $e_j \in R^m$ being the "unit" vector

(58) $$w(t+1) = -\frac{1}{2}H(t+1) \cdot$$

$$\cdot \sum_{j=1}^{m} e_j \cdot \left[tr\left[\hat{M}_j(t+1)\Sigma(t+1 \mid t)\right] + tr\left[\hat{Q}_j(t+1)\Theta(t+1)\right]\right]$$

- State update estimate:

(59) $$\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) +$$

$$+ H(t+1)\left[z(t+1) - h\left(\hat{x}(t+1 \mid t), 0, t+1\right)\right] + w(t+1)$$

The SOF Update-Bias Term

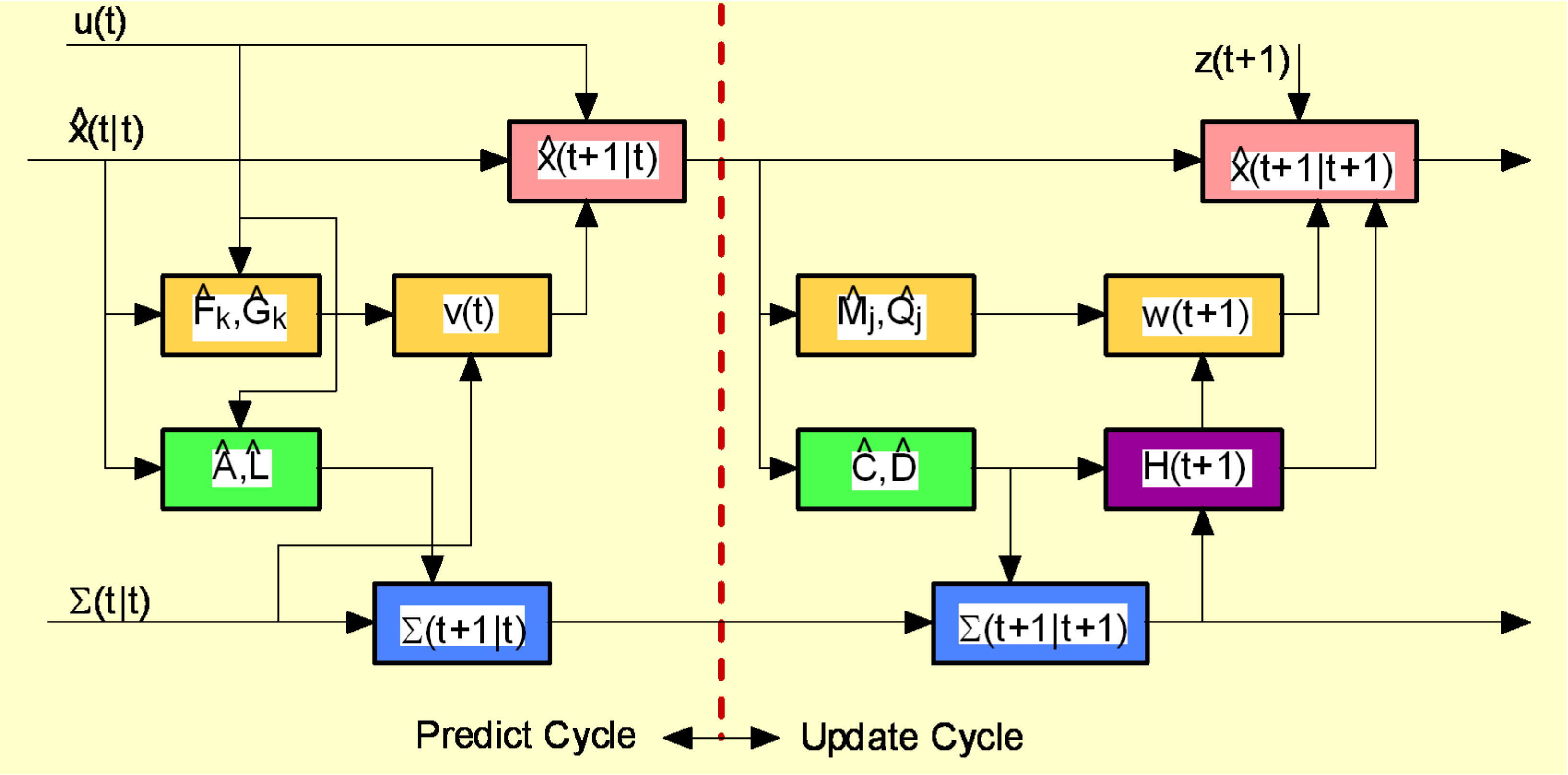
- In the SOF the state-update (59) includes the bias correction term

(60) $$w(t+1) = -\frac{1}{2} H(t+1) \cdot$$

$$\cdot \sum_{j=1}^{m} e_j \cdot \left[tr\left[\hat{M}_j(t+1) \Sigma(t+1 \mid t) \right] + tr\left[\hat{Q}_j(t+1) \Theta(t+1) \right] \right]$$

- This term is significant if at the predicted estimate $\hat{x}(t+1 \mid t)$
  - (a). the sensor nonlinearity $h(x, \theta, t)$ has significant curvature in the $x-$direction, $\hat{M}_j(t+1),$ and large covariance $\Sigma(t+1 \mid t),$ and / or
  - (b). the sensor nonlinearity $h(x, \theta, t)$ has significant curvature in the $\theta-$direction, $\hat{Q}_j(t+1),$ and large sensor-noise covariance $\Theta(t+1)$
- For the SOF, some authors [1], [4], [6] also include another correction term in the updated covariance of eq. (56) and the SOF gain of eq. (57), which involve double-sum terms of the type $tr\left[\hat{M}_i \Sigma \hat{M}_j \Sigma \right]$

Equation Flow-Chart for SOF



Elements of Proof: SOF Predict Cycle, I

- Update estimation error: $\tilde{x}(t \mid t) \equiv x(t) - \hat{x}(t \mid t)$
- Predict estimation error: $\tilde{x}(t+1 \mid t) \equiv x(t+1) - \hat{x}(t+1 \mid t)$

(61) $$x(t+1) = f\big(x(t), u(t), \xi(t), t\big)$$

(62) $$\hat{x}(t+1 \mid t) = f\big(\hat{x}(t \mid t), u(t), 0, t\big) + v(t)$$; for EKF $v(t) = 0$

(63) $$\tilde{x}(t+1 \mid t) = f\big(x(t), u(t), \xi(t), t\big) - f\big(\hat{x}(t \mid t), u(t), 0, t\big) - v(t)$$

- Expand eq. (63) in a Taylor series up to quadratic terms

(64) $$\tilde{x}(t+1 \mid t) \cong \hat{A}(t)\tilde{x}(t \mid t) + \hat{L}(t)\xi(t)$$

$$+\frac{1}{2}\sum_{k=1}^{n} e_k \cdot \tilde{x}'(t \mid t)\hat{F}_k(t)\tilde{x}(t \mid t) + \frac{1}{2}\sum_{k=1}^{n} e_k \cdot \xi'(t)\hat{G}_k(t)\xi(t) +$$

$$+\sum_{k=1}^{n} e_k \cdot \tilde{x}'(t \mid t)\hat{N}_k(t)\xi(t) - v(t)$$

where the Jacobian matrices $\hat{A}(t), \hat{L}(t)$ are given by eqs. (35) and (36) and the Hessian matrices $\hat{F}_k(t), \hat{G}_k(t), \hat{N}_k(t)$ are given by eqs. (41) to (43)

Elements of Proof: SOF Predict Cycle, II

- Take expectations of both sides of eq. (64)

(65) $$E\{\tilde{x}(t+1 \mid t)\} \cong \hat{A}(t)E\{\tilde{x}(t \mid t)\} + \hat{L}(t)E\{\xi(t)\}$$

$$+\frac{1}{2}\sum_{k=1}^{n} e_k \cdot E\{\tilde{x}'(t \mid t)\hat{F}_k(t)\tilde{x}(t \mid t)\} + \frac{1}{2}\sum_{k=1}^{n} e_k \cdot E\{\xi'(t)\hat{G}_k(t)\xi(t)\} +$$

$$+\sum_{k=1}^{n} e_k \cdot E\{\tilde{x}'(t \mid t)\hat{N}_k(t)\xi(t)\} - v(t)$$

- Assume $E\{\tilde{x}(t \mid t)\} = 0,$ and require $E\{\tilde{x}(t+1 \mid t)\} = 0$ to obtain, noting that $\tilde{x}(t \mid t)$ and $\xi(t)$ are independent, and

(66) $$E\{\tilde{x}'(t \mid t)\hat{F}_k(t)\tilde{x}(t \mid t)\} = tr\left[\hat{F}_k(t) \cdot E\{\tilde{x}(t \mid t)\tilde{x}'(t \mid t)\}\right]$$

(67) $$E\{\xi'(t)\hat{G}_k(t)\xi(t)\} = tr\left[\hat{G}_k(t) \cdot E\{\xi(t)\xi'(t)\}\right] \quad \Rightarrow$$

(68) $$v(t) = \frac{1}{2}\sum_{k=1}^{n} e_k tr\left[\hat{F}_k(t)\Sigma(t \mid t)\right] + \frac{1}{2}\sum_{k=1}^{n} e_k tr\left[\hat{G}_k(t)\Xi(t)\right]$$ (for EKF $v(t) = 0$)

Elements of Proof: SOF Update Cycle, I

- Examine the state - update equation for SOF (for EKF $w(t+1)=0$)

(69) $z(t+1) = h\big(x(t+1), \theta(t+1), t+1\big)$

(70) $\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) + H(t+1)\big[z(t+1) - h\big(\hat{x}(t+1 \mid t), 0, t+1\big)\big] + w(t+1)$

$= \hat{x}(t+1 \mid t) + H(t+1)\big[h\big(x(t+1), \theta(t+1), t+1\big) - h\big(\hat{x}(t+1 \mid t), 0, t+1\big)\big] + w(t+1)$

- Let: $\tilde{x}(t+1 \mid t+1) \equiv x(t+1) - \hat{x}(t+1 \mid t+1);\quad \tilde{x}(t+1 \mid t) \equiv x(t+1) - \hat{x}(t+1 \mid t)$
- Expand $h\big(x(t+1), \theta(t+1), t+1\big)$ in a Taylor series through quadratic terms

(71) $h\big(x(t+1), \theta(t+1), t+1\big) - h\big(\hat{x}(t+1 \mid t), 0, t+1\big) =$

$$= \hat{C}(t+1)\tilde{x}(t+1 \mid t) + \hat{D}(t+1)\theta(t+1) + \frac{1}{2}\sum_{j=1}^{m} e_j \tilde{x}'(t+1 \mid t)\hat{M}_j(t+1)\tilde{x}(t+1 \mid t) +$$

$$+\frac{1}{2}\sum_{j=1}^{m} e_j \theta'(t+1)\hat{Q}_j(t+1)\theta(t+1) + \sum_{j=1}^{m} e_j \tilde{x}'(t+1 \mid t)\hat{R}_j(t+1)\theta(t+1)$$

where $\hat{C}(t+1), \hat{D}(t+1)$ are given by eqs. (37) and (38), and $\hat{M}_j(t+1)$, $\hat{Q}_j(t+1), \hat{R}_j(t+1)$ are given by eqs. (45) to (47)

Elements of Proof: SOF Update Cycle, II

- Subtract $x(t+1)$ from both sides of eq. (70), change sign, and substitute the Taylor series expansion (71) into eq. (70)

(72) $\tilde{x}(t+1 \mid t+1) \cong \tilde{x}(t+1 \mid t) - H(t+1)\left[\hat{C}(t+1)\tilde{x}(t+1 \mid t) + \hat{D}(t+1)\theta(t+1)\right]$

$$-H(t+1)\left[\frac{1}{2}\sum_{j=1}^{m} e_j \tilde{x}'(t+1 \mid t)\hat{M}_j(t+1)\tilde{x}(t+1 \mid t) + \frac{1}{2}\sum_{j=1}^{m} e_j \theta'(t+1)\hat{Q}_j(t+1)\theta(t+1)\right]$$

$$-H(t+1)\sum_{j=1}^{m} e_j \cdot \tilde{x}'(t+1 \mid t)\hat{R}_j(t+1)\theta(t+1) - w(t+1)$$ ($w(t+1) = 0$ for the EKF)

- Take expectations of both sides of eq. (72), assume that $E\{\tilde{x}(t+1 \mid t)\} = 0,$ and require that $E\{\tilde{x}(t+1 \mid t+1)\} = 0,$ to obtain the update - bias term

(73) $w(t+1) = -\frac{1}{2} H(t+1) \cdot \sum_{j=1}^{m} e_j \left[tr\left[\hat{M}_j(t+1)\Sigma(t+1 \mid t)\right] + tr\left[\hat{Q}_j(t+1)\Theta(t+1)\right]\right]$

Elements of Proof: Covariances, I

- In the EKF we only include linear terms in the Taylor series expansions. Then the covariance equation includes only expected values of quadratic terms in the estimation error $\tilde{x}(.)$.
- Optimizing the gain matrix $H(t+1)$ to minimize the trace of the error covariance matrix yields the standard formula (51) for the gain matrix and for the covariance propagation equations (49) and (50)

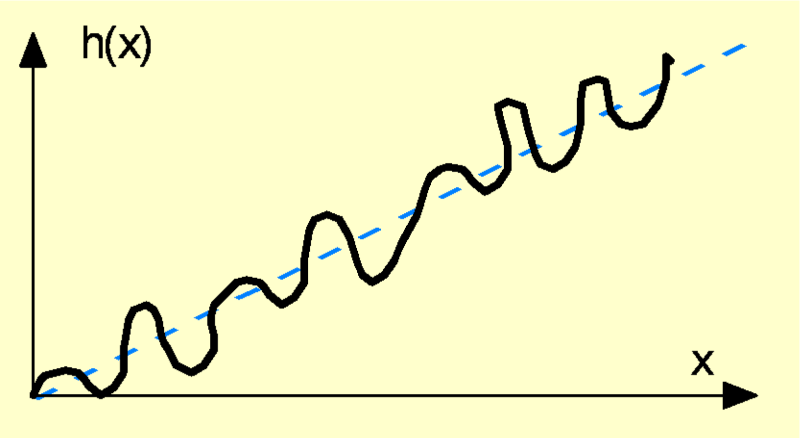
Elements of Proof: Covariances, II

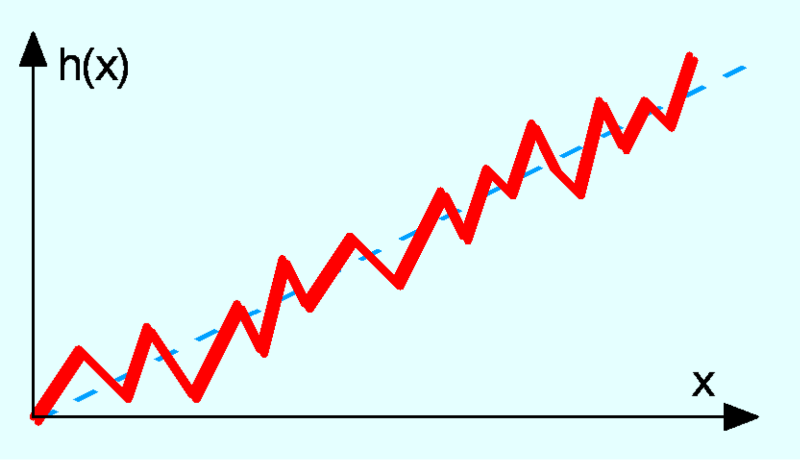
- In the SOF we retain quadratic terms in the Taylor series expansions
- When we calculate the error covariances $\Sigma(.) = E\{\tilde{x}(.)\tilde{x}'(.)\}$ using either the predict error equation (64) or the update error equation (72) we obtain cubic terms that involve expected values of triple $\tilde{x}(.)$ products and quartic terms that involve expected values of quadruple $\tilde{x}(.)$ products
- Some authors, [1], [4], [6], [7], estimate these "extra" expected values, by making the assumption that the estimation error $\tilde{x}(.)$ satisfies an approximate gaussian distribution, because one can calculate its third and fourth moments from the mean and covariance
- Using the above gaussian assumption one obtains additional correction terms (very very complex) in the SOF covariance propagation equations and the SOF gain matrix
- The version of the SOF presented does not include these extra correction terms

Other Nonlinear Filters

- More complicated SOF filters include covariance correction terms in the covariance propagation and filter gain based on approximations to the 3rd and 4th moments of the state-estimation errors in eqs. (64) and (72) ; see [1], [6], [7]
- For very low-order systems one can include the cubic, quartic etc. terms in the Taylor series expansion; see Example 6.2-1 in [1] pp. 209-210
- The "Iterated EKF" algorithm uses iterative methods at each predict and update cycle to improve the linearization accuracy; see [1] pp. 190-192, and [7]
- The "Statistical Linearization" method uses describing function methods to approximate nonlinearities; [1] pp. 204-207

Problems with Taylor Expansions





- For these type of sensors both the EKF and the SOF will have problems, since the local slopes and curvatures are misleading
- It would be better to approximate the sensor nonlinearity $h(x)$ by a straight line, and increase slightly the covariance of the sensor noise

Concluding Remarks

- The EKF and SOF algorithms have been extensively used in numerous applications
    - there is no *a-priori* guarantee that their performance will be satisfactory
- They may even diverge, when the state uncertainty is sufficiently large so that the local linearizations (Jacobian matrices) and curvature estimates (Hessian matrices) are evaluated at state-estimates very far from the true state
- The so-called "Gaussian Sum" nonlinear filter can be used whenever the standard EKF and SOF may diverge
    - the Gaussian Sum (GS) method employs parallel banks of EKFs (or SOFs)
    - the GS filter computational requirements are high
    - we shall discuss the GS filter in the sequel

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