Optimal Linear Smoothing Algorithms for Discrete-Time Systems

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Last Revised: October 20, 2001

Ref. No. KF#16

Smoothing (or Interpolation) Algorithms

- A smother estimates the state of a dynamic system at some time *t*, using measurements made both before and after the specific time *t*
- Contrast this to a filter (or a predictor) that uses only past and present measurements at time *t*, to estimate the state at time *t* (or predict the state at some future time, *t*+*T*)
- The accuracy of a smoother state-estimate must be better than its filter estimate, since more measurements are utilized by the smoother
- Often, smoothing algorithms are called "non-causal", since future measurements are used to estimate past states

Types of Smoothers

- Fixed-Interval Smoothers: These use all measurements over a fixed interval to estimate the state at all times in the same interval
- Fixed-Point Smoothers: These are used to estimate the state at some fixed time in the past, given the measurements up to the present time, and updating that state-estimate every time a new measurement is made
- Fixed-Lag Smoothers: These estimate the state at a fixed time-lag from the present measurement, and update the state-estimate (using the same fixed lag) every time a new measurement is made

Why Smoothing (or Interpolation)?

- Fixed-Interval (FI) smoothers are used for post-processing of experimental data to provide the most accurate state-estimate
  - useful in trouble-shooting, e.g. post-processing of tracking radar measurements for an experimental missile to check that the guidance system has worked as designed
- Fixed-Point (FP) smoothers are used when one is only interested at a most-accurate state-estimate at a specific time instant
  - often is used in obtaining a very accurate estimate of the initial state, e.g. useful in determining orbit injection parameters for an orbiting satellite
- Fixed-Lag (FL) smoothers are used whenever a delay in obtaining a more accurate state-estimate can be tolerated
  - sometimes can be used in conjunction with telemetry data

Topic Outline and Remarks

- We summarize the complete equations for all three smoothing algorithms for the general linear time-varying (LTV), discrete-time case with gaussian uncertainties, following Ref. [1]
  - for simplicity, we do not include deterministic inputs
  - results are also available for continuous-time LTV problems, see [1]-[5], but are not presented here
  - results are also available for (suitably formulated) continuous-time LTV dynamics with discrete-time measurements
- No detailed proofs are provided. They hinge on showing that the appropriate conditional probability density function of the state is gaussian, so that it can be fully described by its conditional mean and covariance (whose equations we present, [1])
- All three smoothing algorithms require the implementation of the discrete-time Kalman filter (predict and update cycles) for the filter state-estimate and associated covariance matrices

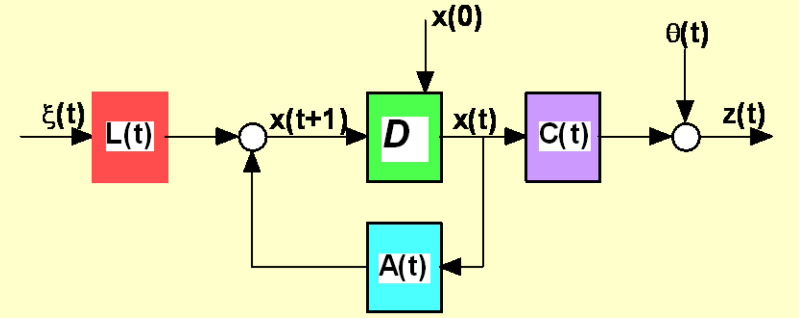
Definitions

LTV DISCRETE- TIME MODEL

$$t = 0, 1, 2, 3, \dots$$

(1) $x(t+1) = A(t)x(t) + L(t)\xi(t)$

$z(t+1) = C(t+1)x(t+1) + \theta(t+1)$



UNCERTAINTY

- Initial state $x(0) \sim N(\bar{x}_0, \Sigma_0)$
- Plant noise $\xi(t) \sim N(0, \Xi(t)\delta_{t\tau})$
- Sensor noise $\theta(t) \sim N(0, \Theta(t)\delta_{t\tau})$
- $x(0), \xi(t), \theta(\sigma)$ independent $\forall t, \sigma$
- Assume that all standard KF variables are available, $\hat{x}(t \mid t)$, $\Sigma(t \mid t)$, $\hat{x}(t+1 \mid t)$, $\Sigma(t+1 \mid t)$, $\hat{x}(t+1 \mid t+1)$, $\Sigma(t+1 \mid t+1)$

- Data set, $Z(t)$

(2) $Z(t) = \{z(1), \dots, z(t)\}$

- In smoothing problems we want, for $\tau \leq t$

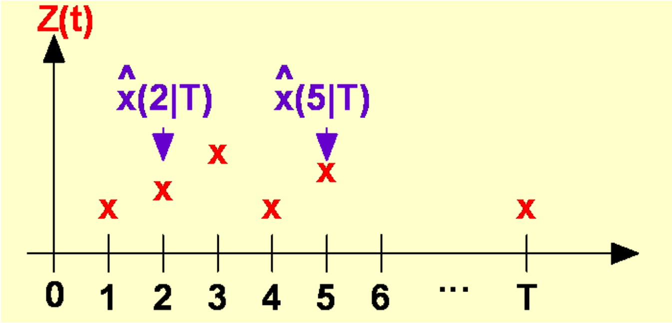
(3) $\hat{x}(\tau \mid t) \equiv E\{x(\tau) \mid Z(t)\}$

(4) $\Sigma(\tau \mid t) \equiv cov[x(\tau); x(\tau) \mid Z(t)]$

Fixed-Interval Smoothing: Definition

- Given a fixed number of measurements (T is fixed), $Z(T)$
- Determine for all $t,\ 0 \le t \le T,$
  - the conditional mean, $\hat{x}(t \mid T)$
  - the conditional covariance, $\Sigma(t \mid T)$
- In example shown,

  $t = 2, \quad t = 5$



Fixed-Point Smoothing: Definition

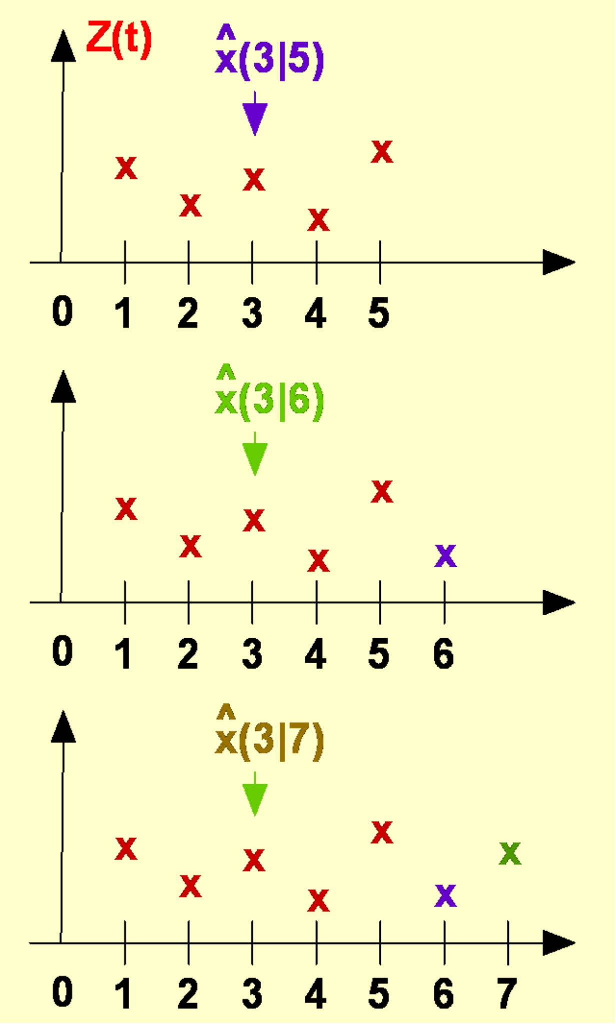
- Estimate - time, τ is fixed
- Given data $Z(t), t \geq \tau$
- Determine $\hat{x}(\tau \mid t)$
- Determine $\Sigma(\tau \mid t)$

- In example shown

$\tau = 3, \quad t = 5$

$\tau = 3, \quad t = 6$

$\tau = 3, \quad t = 7$



Fixed-Lag Smoothing: Definition

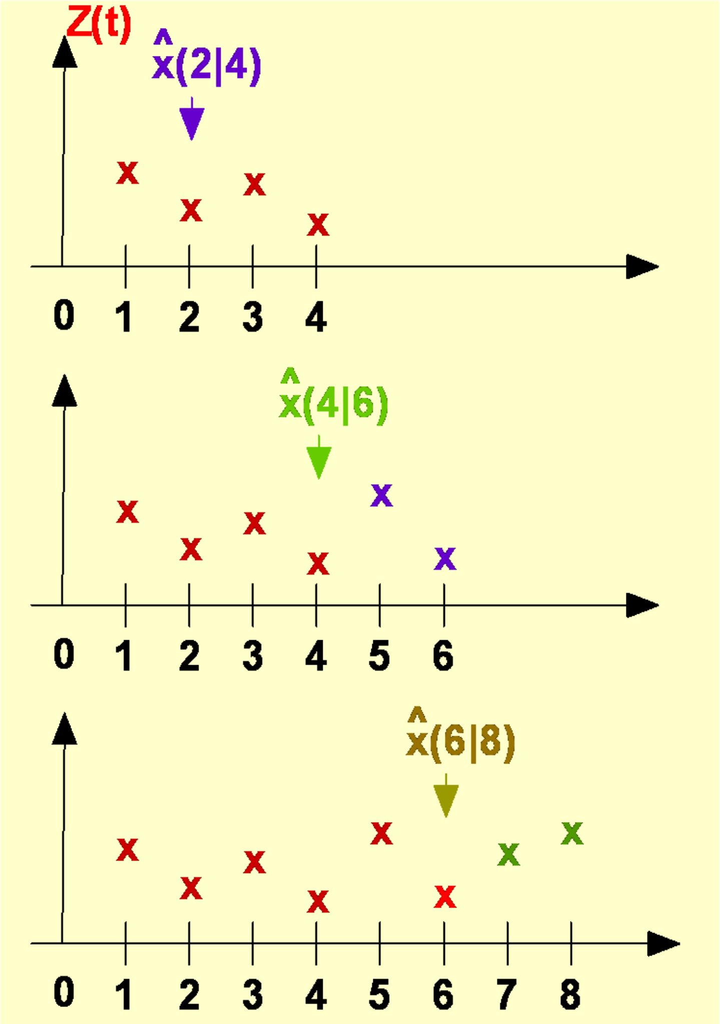
- Lag k is fixed
- Given $Z(t+k)$, for $t > k$
- Find $\hat{x}(t \mid t+k)$
- Find $\Sigma(t \mid t+k)$

- In example shown, $k = 2$

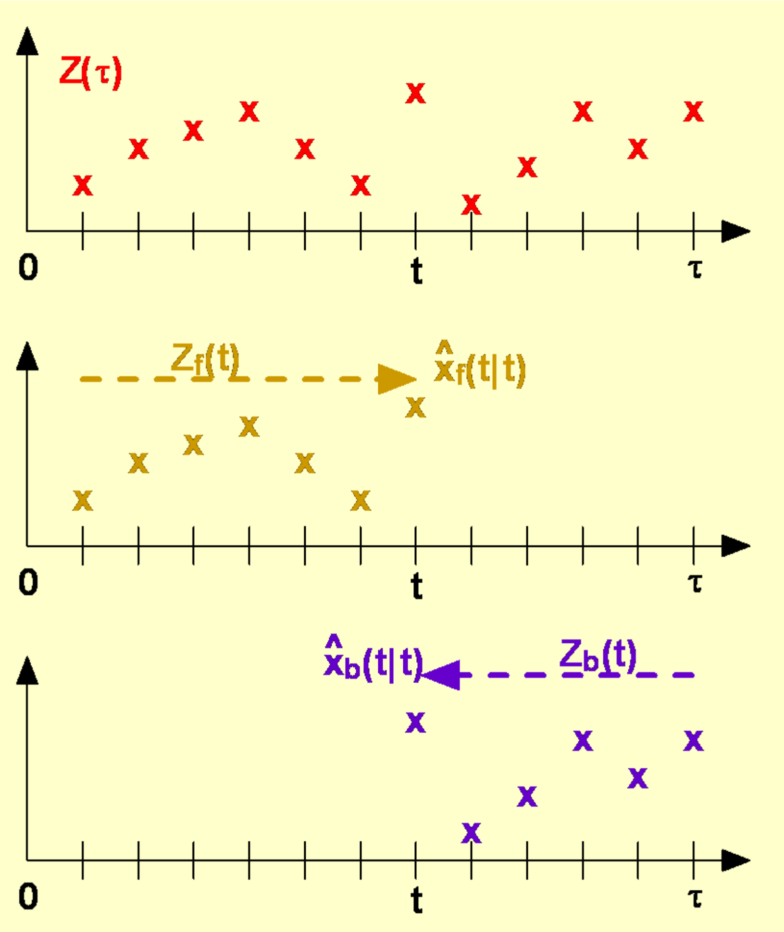
  $t = 2, \quad t+k = 4$

  $t = 4, \quad t+k = 6$

  $t = 6, \quad t+k = 8$



Smoothing: Basic Ideas

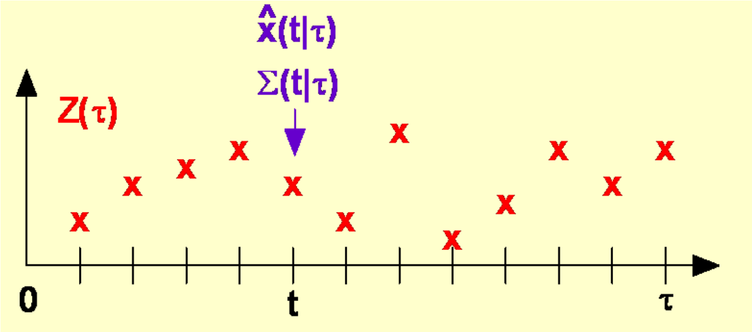


- Start with set of data $Z(\tau)$
- For any time $t, 0 \le t \le \tau,$ run ordinary KF "forward-in-time" to obtain state-estimate
$$\hat{x}_f(t \mid t) = E\{x(t) \mid Z_f(t)\}$$
- For any time $t, 0 \le t \le \tau,$ run ordinary KF "backward-in-time" to obtain state-estimate
$$\hat{x}_b(t \mid t) = E\{x(t) \mid Z_b(t)\}$$
- Determine the optimal way of combining the estimates $\hat{x}_f(t \mid t)$ and $\hat{x}_b(t \mid t)$ to obtain
$$\hat{x}(t \mid \tau) = E\{x(t) \mid Z(\tau)\}$$

Fixed-Interval Smoother: Summary

NOTATION

- $\hat{x}(t \mid t), \hat{x}(t+1 \mid t)$ ordinary KF state-estimates
- $\Sigma(t \mid t), \Sigma(t+1 \mid t)$ ordinary KF covariances



- Given data $Z(\tau), 0 \le t \le \tau$
- The smoother conditional pdf

  $p(x(t) \mid Z(\tau))$ is gaussian

  with conditional mean

(5) $\hat{x}(t \mid \tau) = E\{x(t) \mid Z(\tau)\}$

and conditional covariance

(6) $\Sigma(t \mid \tau) = cov[x(t); x(t) \mid Z(\tau)]$

- Smoother state-estimate, $\hat{x}(t \mid \tau)$

(7) $\hat{x}(t \mid \tau) = \hat{x}(t \mid t) + G(t)[\hat{x}(t+1 \mid \tau) - \hat{x}(t+1 \mid t)]$

(8) boundary condition: $\hat{x}(\tau \mid \tau)$ from KF

- Smoother gain matrix, $G(t)$

(9) $G(t) = \Sigma(t \mid t) A'(t) \Sigma^{-1}(t+1 \mid t)$

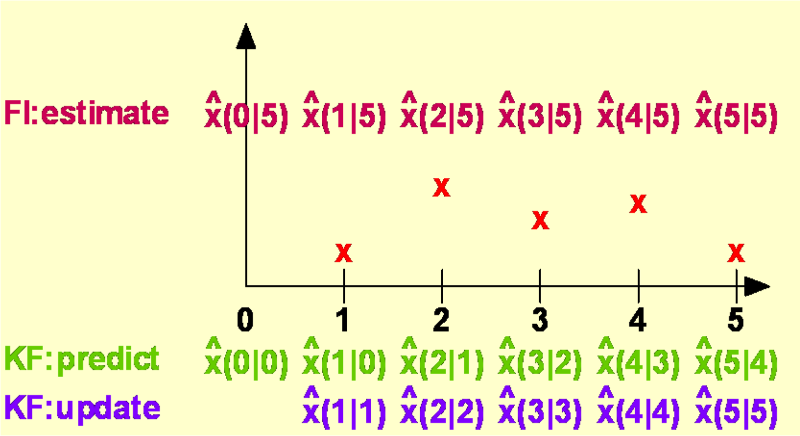
- Smoother covariance, $\Sigma(t \mid \tau)$

(10) $\Sigma(t \mid \tau) = \Sigma(t \mid t) + G(t)[\Sigma(t+1 \mid \tau) - \Sigma(t+1 \mid t)]G'(t)$

(11) boundary condition: $\Sigma(\tau \mid \tau)$ from KF

- NOTE: Both $\hat{x}(t \mid \tau)$ and $\Sigma(t \mid \tau)$ are computed recursively, "backward-in-time"

Fixed-Interval Smoother: Example



- Data: $Z(5) = \{z(1), z(2), \ldots, z(5)\}$
- Run KF forward in time to generate

$\hat{x}(1 \mid 0), \hat{x}(1 \mid 1), \hat{x}(2 \mid 1), \hat{x}(2 \mid 2), \ldots, \hat{x}(5 \mid 5)$

$\Sigma(1 \mid 0), \Sigma(1 \mid 1), \Sigma(2 \mid 1), \Sigma(2 \mid 2), \ldots, \Sigma(5 \mid 5)$

FI Smoother State Estimates $\hat{x}(t \mid 5)$ are computed bacward in time

$$\hat{x}(4 \mid 5) = \hat{x}(4 \mid 4) + G(4)[\hat{x}(5 \mid 5) - \hat{x}(5 \mid 4)]$$
$$\hat{x}(3 \mid 5) = \hat{x}(3 \mid 3) + G(3)[\hat{x}(4 \mid 5) - \hat{x}(4 \mid 3)]$$
$$\hat{x}(2 \mid 5) = \hat{x}(2 \mid 2) + G(2)[\hat{x}(3 \mid 5) - \hat{x}(3 \mid 2)]$$
$$\hat{x}(1 \mid 5) = \hat{x}(1 \mid 1) + G(1)[\hat{x}(2 \mid 5) - \hat{x}(2 \mid 1)]$$
$$\hat{x}(0 \mid 5) = \hat{x}(0 \mid 0) + G(0)[\hat{x}(1 \mid 5) - \hat{x}(1 \mid 0)]$$

FI Smoother Gain Calculation

$$G(0) = \Sigma(0 \mid 0) A'(0) \Sigma^{-1}(1 \mid 0)$$
$$G(1) = \Sigma(1 \mid 1) A'(1) \Sigma^{-1}(2 \mid 1)$$
$$G(2) = \Sigma(2 \mid 2) A'(2) \Sigma^{-1}(3 \mid 2)$$
$$G(3) = \Sigma(3 \mid 3) A'(3) \Sigma^{-1}(4 \mid 3)$$
$$G(4) = \Sigma(4 \mid 4) A'(4) \Sigma^{-1}(5 \mid 4)$$

Fixed-Point Smoothing: Summary

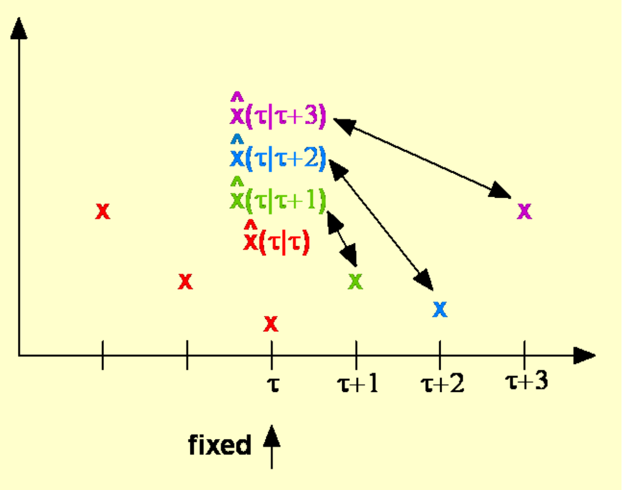
- Given: τ = fixed, growing data set $Z(t),\ t > \tau$

(12) $t = \tau + 1,\ \tau + 2,\ \tau + 3, \dots$

- Smoother state-estimate, $\hat{x}(\tau \mid t)$, is computed forward-in-time for each new measurement

(13) $\hat{x}(\tau \mid t) = \hat{x}(\tau \mid t-1) + D(\tau \mid t)\left[\hat{x}(t \mid t) - \hat{x}(t \mid t-1)\right]$

(14) Boundary condition: $\hat{x}(\tau \mid \tau)$ from KF



- Smoother gain: $D(\tau \mid t)$, is calculated forward-in-time (precomputed)

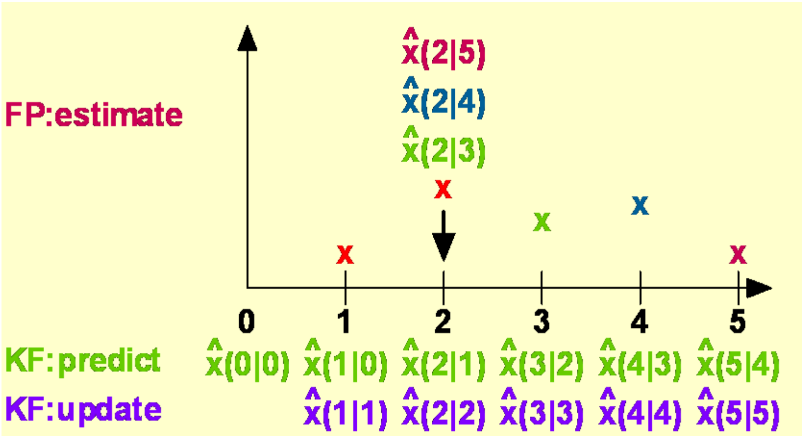
(15) $D(\tau \mid t) = D(\tau \mid t-1)\Sigma(t-1 \mid t-1)A'(t-1)\Sigma^{-1}(t \mid t-1); \quad D(\tau \mid \tau) = I$

- Smoother covariance: $\Sigma(\tau \mid t)$, is also calculated forward-in-time

(16) $\Sigma(\tau \mid t) = \Sigma(\tau \mid t-1) + D(\tau \mid t)\left[\Sigma(\tau \mid \tau) - \Sigma(\tau \mid \tau - 1)\right]D'(\tau \mid t);$

where $\Sigma(\tau \mid \tau)$ and $\Sigma(\tau \mid \tau - 1)$ from KF

Fixed-Point Smoothing: Example



- Data: $Z(5)=\{z(1),z(2),\ldots,z(5)\}$
- Desired FP estimate at $\tau=2,\ \hat{x}(2\mid t)$
- Run KF forward in time to generate

$\hat{x}(1\mid 0),\hat{x}(1\mid 1),\hat{x}(2\mid 1),\hat{x}(2\mid 2),\ldots,\hat{x}(5\mid 5)$

$\Sigma(1\mid 0),\Sigma(1\mid 1),\Sigma(2\mid 1),\Sigma(2\mid 2),\ldots,\Sigma(5\mid 5)$

FP Smoother estimate, $\hat{x}(2\mid t)$, computed forward-in-time, for each new measurement $\hat{x}(2\mid 2)$ from KF

$$\hat{x}(2\mid 3)=\hat{x}(2\mid 2)+D(2\mid 3)[\hat{x}(3\mid 3)-\hat{x}(3\mid 2)]$$

$$\hat{x}(2\mid 4)=\hat{x}(2\mid 3)+D(2\mid 4)[\hat{x}(4\mid 4)-\hat{x}(4\mid 3)]$$

$$\hat{x}(2\mid 5)=\hat{x}(2\mid 4)+D(2\mid 5)[\hat{x}(5\mid 5)-\hat{x}(5\mid 4)]$$

FP Smoother Gain $D(2\mid t)$

$$D(2\mid 2)=I$$

$$D(2\mid 3)=D(2\mid 2)\Sigma(2\mid 2)A'(2)\Sigma^{-1}(3\mid 2)$$

$$D(2\mid 4)=D(2\mid 3)\Sigma(3\mid 3)A'(3)\Sigma^{-1}(4\mid 3)$$

$$D(2\mid 5)=D(2\mid 4)\Sigma(4\mid 4)A'(4)\Sigma^{-1}(5\mid 4)$$

Fixed-Point Smoothing: Remarks

- Given fixed time, $\tau,$ and growing data set, $Z(t), t > \tau$
- The conditional pdf of the state, $p(x(\tau) \mid Z(t)),$ is gaussian with conditional mean generated by eq. (13) and conditional covariance generated by eq. (15). Thus, in eq. (13) the fixed-point state estimate is

(17) $\hat{x}(\tau \mid t) \equiv E\{x(\tau) \mid Z(t)\}$

and in eq. (15) the fixed-point covariance matrix is

(18) $\Sigma(\tau \mid t) \equiv \mathrm{cov}[x(\tau); x(\tau) \mid Z(t)]$

Fixed-Lag Smoothing: Summary

- Given growing data set $Z(t);\ t = 1, 2, 3, \dots$
- Given fixed lag $k \geq 1$
- Smoother state estimate: $\hat{x}(t+1 \mid t+1+k)$

(19) $\hat{x}(t+1 \mid t+1+k) = A(t)\hat{x}(t \mid t+k) +$

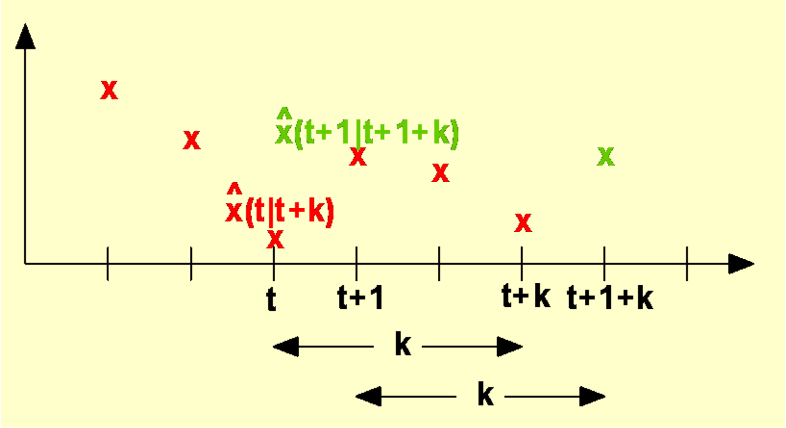
$+ L(t)\Xi(t)L'(t)A'(t)\Sigma^{-1}(t \mid t)\left[\hat{x}(t \mid t+k) - \hat{x}(t \mid t)\right] +$

$+ D(t+1 \mid t+1+k)H(t+1+k)\cdot$

$\cdot\left[z(t+1+k) - C(t+1+k)A(t+k)\hat{x}(t+k \mid t+k)\right]$

with $\hat{x}(0 \mid k)$ from fixed-point smoother, and

$H(t+1+k)$ is the KF gain matrix



(20) $D(t+1 \mid t+1+k) = G^{-1}(t)D(t \mid t+k)G(t+k)$

with boundary condition $D(0 \mid k)$ from fixed-point smoother, and $G(t)$ being the fixed-interval smoother gain matrix, eq. (9), $G(t) = \Sigma(t \mid t)A'(t)\Sigma^{-1}(t+1 \mid t)$

- Covariance matrix, $\Sigma(t+1 \mid t+1+k)$

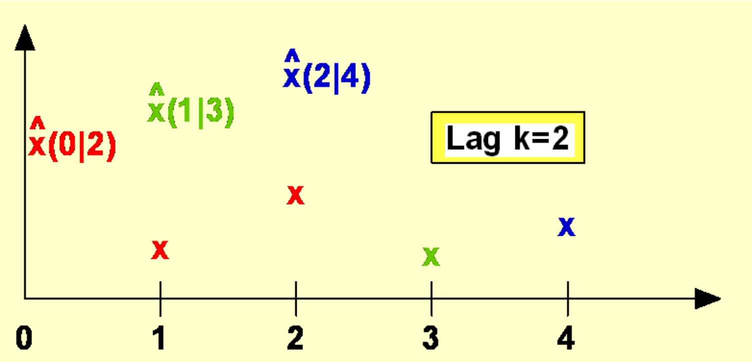
(21) $\Sigma(t+1 \mid t+1+k) = \Sigma(t+1 \mid t)$

$- D(t+1 \mid t+1+k)H(t+1+k)C(t+1+k)\Sigma(t+1+k \mid t+k)D'(t+1 \mid t+1+k)$

$- G^{-1}(t)\left[\Sigma(t \mid t) - (\Sigma(t \mid t+k)\right]G'^{-1}(t)$

with boundary condition $\Sigma(0 \mid k)$ from fixed-point smoother

Fixed-Lag Smoothing: Example



- Data: $Z(4) = \{z(1), z(2), \ldots, z(4)\}$
- Fixed lag: $k = 2$
- Desired FL estimates at $t = 0,1,2$
- Run KF forward in time to generate

$\hat{x}(1 \mid 0), \hat{x}(1 \mid 1), \hat{x}(2 \mid 1), \hat{x}(2 \mid 2), \ldots, \hat{x}(4 \mid 4)$

$\Sigma(1 \mid 0), \Sigma(1 \mid 1), \Sigma(2 \mid 1), \Sigma(2 \mid 2), \ldots, \Sigma(4 \mid 4)$

FL smoother state-estimate

$\hat{x}(0 \mid 2)$ from fixed-point smoother

$$\hat{x}(1 \mid 3) = A(0)\hat{x}(0 \mid 2) + L(0)\Xi(0)L'(0) \cdot A'^{-1}(0)\Sigma^{-1}(0 \mid 0)[\hat{x}(0 \mid 2) - \hat{x}(0 \mid 0)] + D(1 \mid 3)H(3)[z(3) - C(3)\hat{x}(3 \mid 2)]$$

$$\hat{x}(2 \mid 4) = A(1)\hat{x}(1 \mid 3) + L(1)\Xi(1)L'(1) \cdot A'^{-1}(1)\Sigma^{-1}(1 \mid 1)[\hat{x}(1 \mid 3) - \hat{x}(1 \mid 1)] + D(2 \mid 4)H(4)[z(4) - C(4)\hat{x}(4 \mid 3)]$$

$D(0 \mid 2)$ from fixed-point smoother

$$D(1 \mid 3) = G^{-1}(0)D(0 \mid 2)G(2)$$

$$G^{-1}(0) = \left[\Sigma(0 \mid 0)A'(0)\Sigma^{-1}(1 \mid 0)\right]^{-1}$$

$$G(2) = \left[\Sigma(2 \mid 2)A'(2)\Sigma^{-1}(3 \mid 2)\right]$$

$$D(2 \mid 4) = G^{-1}(1)D(1 \mid 3)G(3)$$

$$G^{-1}(1) = \left[\Sigma(1 \mid 1)A'(1)\Sigma^{-1}(2 \mid 1)\right]^{-1}$$

$$G(3) = \left[\Sigma(3 \mid 3)A'(3)\Sigma^{-1}(4 \mid 3)\right]$$

Fixed-Lag Smoothing: Remarks

- Given fixed lag, $k \geq 1$, and growing data set, $Z(t)$
- The conditional pdf of the state, $p(x(t) \mid Z(t+k))$, is gaussian with conditional mean generated by eq. (19) and conditional covariance generated by eq. (21). Thus, in eq. (19) the fixed-lag state estimate is

(22) $\hat{x}(t \mid t+k) \equiv E\{x(t) \mid Z(t+k)\}$

and is computed forward-in-time as new measurements are obtained. In eq. (21) the fixed-lag covariance matrix is

(23) $\Sigma(t \mid t+k) \equiv \text{cov}[x(t); x(t) \mid Z(t+k)]$

Remarks

- The very complicated nature of the smoothing algorithms has forced several authors to use different substitutions and formulas
  - be careful when studying the literature: two different authors may present “different looking” algorithms that solve the same smoothing problem
  - we follow [1], Chapter 5, with minor variations
  - this is especially true for the “hairy” equations that characterize the fixed-lag smoother
- The three types of smoothing algorithms have been derived for continuous-time problems, described by stochastic differential equations, using the Kalman-Bucy filter as the starting point; see [1] and [4]
- In principle, one should be able to extend the MMAE framework to the smoothing case. I am not sure that anyone has done that

Concluding Remarks

- Smoothing (or interpolation) algorithms are very useful in several applications, where we can afford "non-real time" (or non-causal) state-estimates in return for increased accuracy over those generated by the corresponding Kalman filter
- As long as the plant white noise, $\xi(t)$, excites all the state variables, smoothing algorithms are guaranteed to yield better accuracy (smaller covariance) than the corresponding Kalman filter algorithm
    - see the discussion on "smoothability" in [1], p. 163
- Smoothing algorithms can be adapted to nonlinear nongaussian problems to yield suboptimal state-estimates

References

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