Random Processes and Linear Systems

MICHAEL ATHANS

MIT & ISR/IST

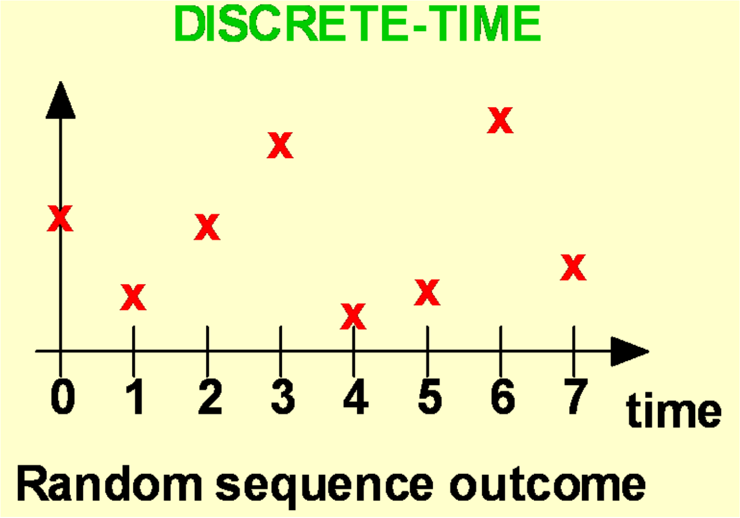
Last Revised: October 28, 2001

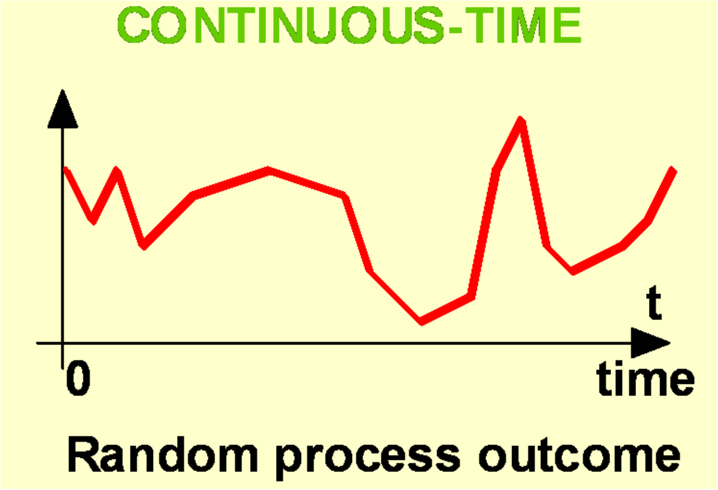
Ref. No. KF #2A

Theme

- Define the concept of continuous-time random processes
  - scalar-valued random processes
  - vector-valued random processes
- Discuss nonstationary and stationary situations, the nature of the probability density functions (pdf), mean and covariance
  - nonstationary means that statistics change with time
  - stationary means that statistics are constant over time
- For stationary random processes we define the "autocorrelation function" and the "power spectral density"
- Demonstrate how to analyze linear time-invariant (LTI) systems driven by stationary random processes
- Define and discuss a modeling tool, the continuous-time white noise process

What Do We Observe?

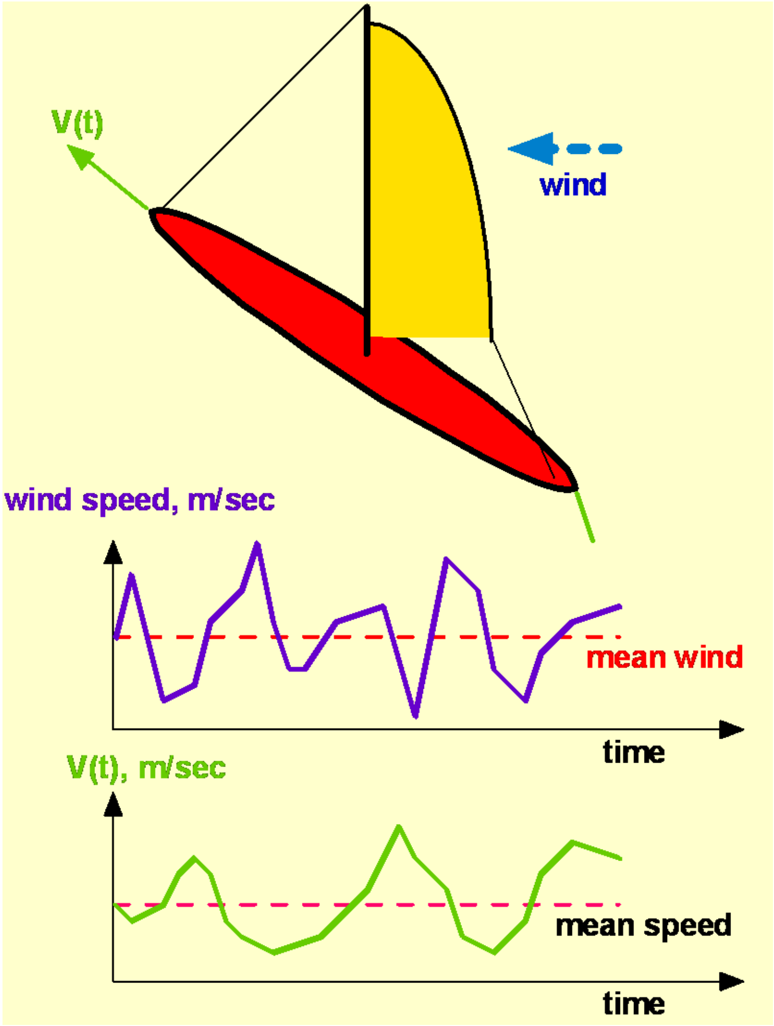




- Example of random sequence: The numerical outcome of sequentially throwing a die, i.e. {3, 6, 6, 2, 3, 1, 2, 5, 5, 5, …}
- Example of a random process: Wind disturbances acting on physical systems
- We concentrate on continuous-time random processes (RPs)
- We shall examine random sequences later

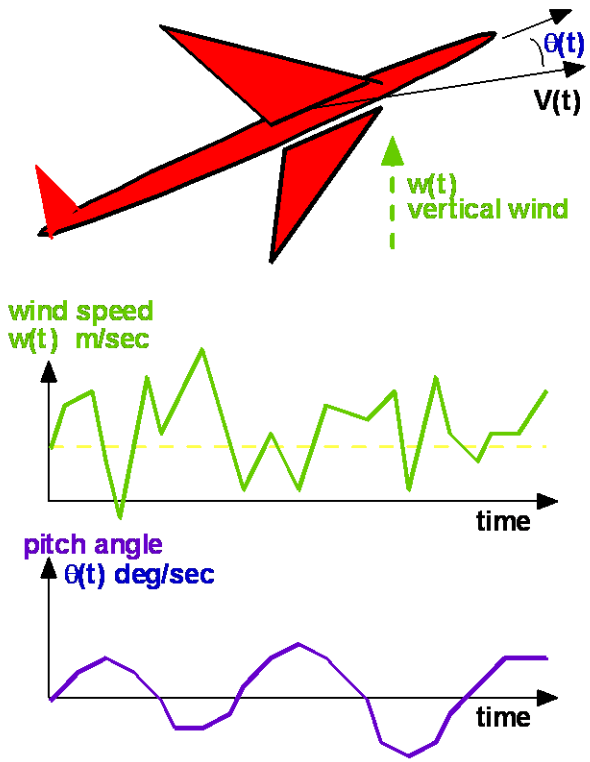
Example: Sailboat Motion

- Wind speed is an example of a random process. There are random wind speed variations about the mean wind speed.
- The wind speed will influence the speed of the sailboat, so that its velocity will also be a random process
- The sailboat speed will depend on the sailboat dynamics and the randomness of the wind speed

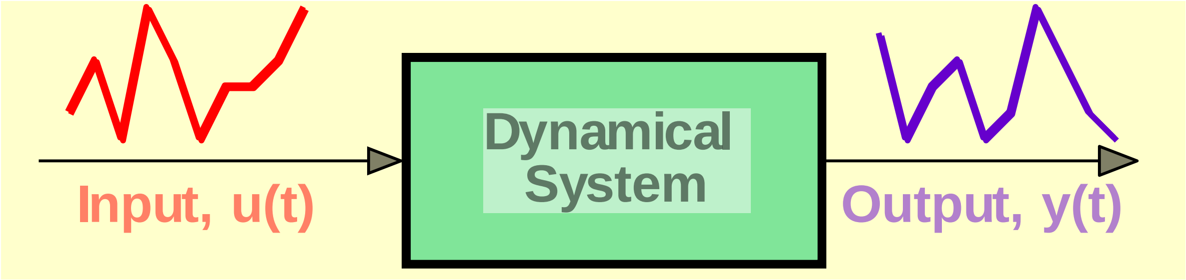


Example: Aircraft Pitching

- Vertical wind gusts are an example of a random process
- Resulting aircraft pitch angle is also a random process
- Aircraft pitch angle depends on aircraft dynamics influenced by the vertical wind gusts



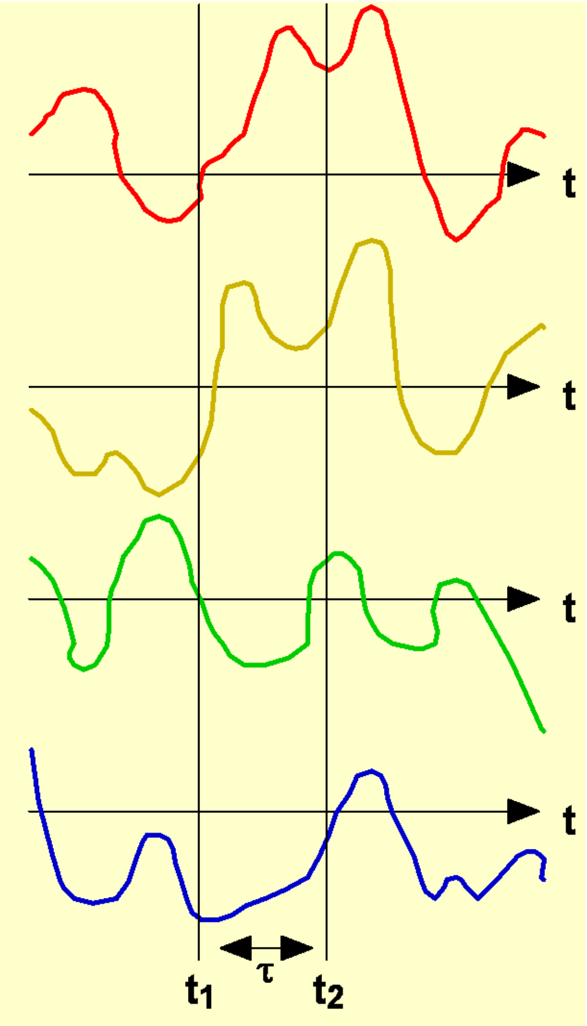
Dynamical Systems with Random Inputs



- We will study how dynamic systems behave when their input, *u(t),* is a random process
- We should expect that the output, *y(t),* will also be a random process
- We must learn how to characterize, in a mathematical framework, random processes
- We must discover how dynamic systems interact with their input random processes and how they generate their output random processes

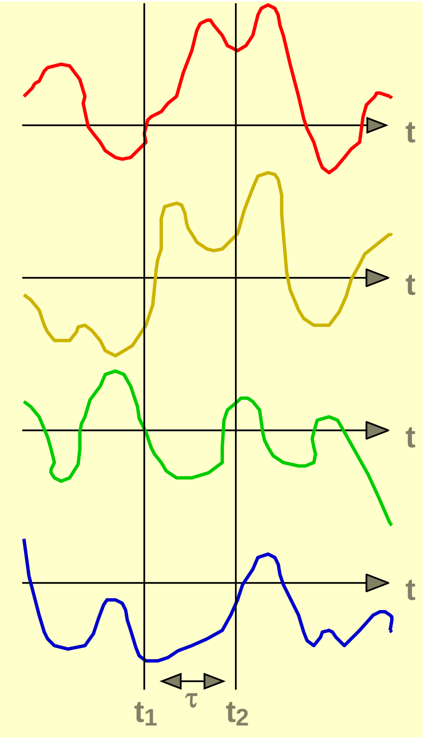
Continuous-Time Random Processes

- Think of a random process (RP) as a collection, or *ensemble*, of time-functions, any one of which may be observed on any trial of an experiment
- Denote the ensemble of functions by $\{x(t)\}$, and of any observed member of the ensemble by $x(t)$
- On repeated trials of experiment, say at times t_1 and t_2, $x(t_1)$ and $x(t_2)$ are random variables
- Example: the RP may represent the temperature from 9:00 to 10:00 am, on July 13, in Boston (different temperature observed each year)



Stationary Random Processes

- At time $t = t_1$: random variable $x(t_1) = x_1$, with pdf $p(x_1, t_1)$
- At time $t = t_2$: random variable $x(t_2) = x_2$, with pdf $p(x_2, t_2)$
- If the statistical properties of the ensemble $\{x(t)\}$ change with time, then we call the random process "non-stationary", and we write the pdf as $p(x(t), t)$
- If the statistical properties of the ensemble $\{x(t)\}$ do not change with time, then we call the random process "stationary", and we write the pdf as $p(x(t))$



Illustration

- Non-stationary random process: the temperature profile, in Boston, on November 28 from 3:00 am to 11:00 pm (it will depend on the time)
- Stationary random process: the temperature profile on November 16, in Boston, from 9:00 to 10:00 am

Statistics of Random Processes

NONSTATIONARY RANDOM PROCESS

- Time - varying mean: $m(t)$

$$m(t) \equiv E\{x(t)\} = \int x(t)\, p(x(t), t)\, dx(t)$$

- Time - varying variance: $\sigma^2(t)$

$$\sigma^2(t) \equiv E\left\{(x(t) - m(t))^2\right\} = \int (x(t) - m(t))^2 p(x(t), t)\, dx(t)$$

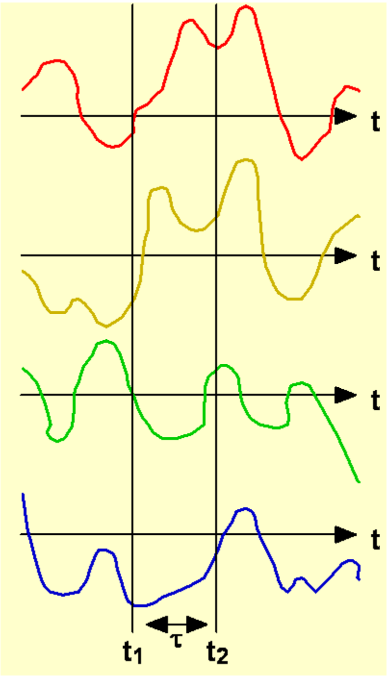
STATIONARY RANDOM PROCESS

- Constant mean: m

$$m \equiv E\{x(t)\} = \int x(t)\, p(x(t))\, dx(t)$$

- Constant variance: σ^2

$$\sigma^2 \equiv E\left\{(x(t) - m)^2\right\} = \int (x(t) - m)^2 p(x(t))\, dx(t)$$



Nonstationary Correlation Function

NONSTATIONARY RANDOM PROCESS, $x(t)$

- Time - varying pdf: $p(x(t), t)$
- Assume: $E\{x(t)\} = 0 \quad \forall t$
- Consider: $x_1 \equiv x(t_1),\ x_2 \equiv x(t_2)$
- The two RVs x_1 and x_2 have a joint density function

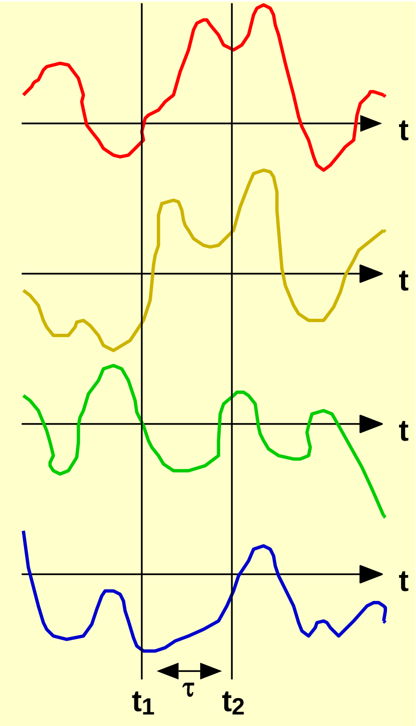
$$p(x_1, x_2) = p(x(t_1), t_1; x(t_2), t_2)$$

- The autocorrelation function $\psi_{xx}(t_1, t_2)$ is defined by

$$\psi_{xx}(t_1, t_2) \equiv E\{x(t_1)x(t_2)\} =$$

$$= \int\int x(t_1)x(t_2)p(x(t_1), t_1; x(t_2), t_2)dx(t_1)dx(t_2)$$

- Note that $\psi_{xx}(t_1, t_2)$ will depend on the absolute values of time, t_1 and t_2



Stationary Autocorrelation Function

STATIONARY RANDOM PROCESS, $x(t)$

- Time-independent pdf: $p(x(t))$
- Assume: $E\{x(t)\} = 0 \quad \forall t$
- Let $t_2 = t_1 + \tau$, and consider $x_1 \equiv x(t_1),\ x_2 \equiv x(t_1 + \tau)$
- The two RVs x_1 and x_2 have a joint density function

$$p(x_1, x_2, \tau) = p(x(t_1), t_1; x(t_1 + \tau), t_1 + \tau) = p(x(t_1), x(t_1 + \tau), \tau)$$

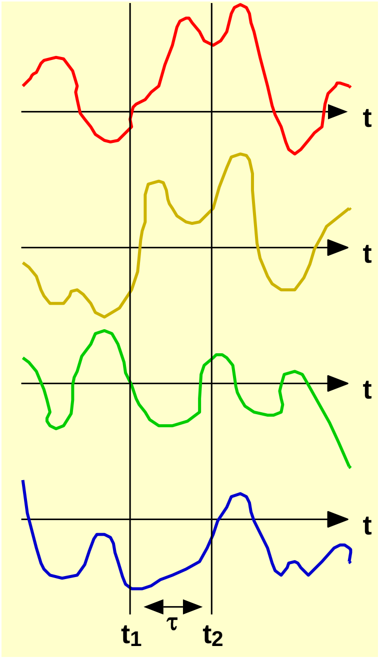
which now only depends on the time-difference τ

- The autocorrelation function $\psi_{xx}(\tau)$ is defined by

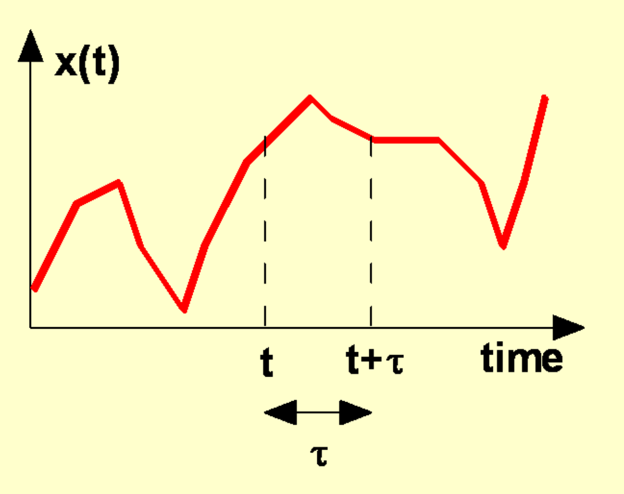
$$\psi_{xx}(\tau) \equiv E\{x(t)x(t+\tau)\} =$$

$$= \int\int x(t)x(t+\tau)p(x(t), x(t+\tau), \tau)dx(t)dx(t+\tau)$$

- Note that in stationary random processes $\psi_{xx}(\tau)$ will only depend on the time-interval τ and not on the absolute value of time t



Autocorrelation Function



- Stationary random process, $x(t)$
- Mean: $E\{x(t)\} = \bar{x} =$ constant for all t

  Assume $\bar{x} = 0$ for convinience
- Variance: $E\left\{x^2(t)\right\} = \sigma_{xx}^2 =$ constant for all t

DEFINITION

- Autocorrelation function of $x(t)$: $\psi_{xx}(\tau)$

  $$\psi_{xx}(\tau) \equiv E\{x(t)x(t+\tau)\}$$
- Autocorrelation function depends only on interval τ and not on time t

PROPERTIES

- $\psi_{xx}(\tau)$ is symmetric, i.e.

  $$\psi_{xx}(\tau) = \psi_{xx}(-\tau)$$
- $\psi_{xx}(0) = \sigma_{xx}^2$

Power Spectral Density (PSD) Function

- Given, zero-mean stationary random process, $x(t)$, with autocorrelation function $\psi_{xx}(\tau)$

DEFINITION

- The power spectral density (PSD) function $\phi_{xx}(\omega)$ of $x(t)$ is the Fourier transform of the autocorrelation function $\psi_{xx}(\tau)$

$$\phi_{xx}(\omega) \equiv \int_{-\infty}^{\infty} \psi_{xx}(\tau) e^{-j\omega\tau} d\tau$$

PSD PROPERTIES

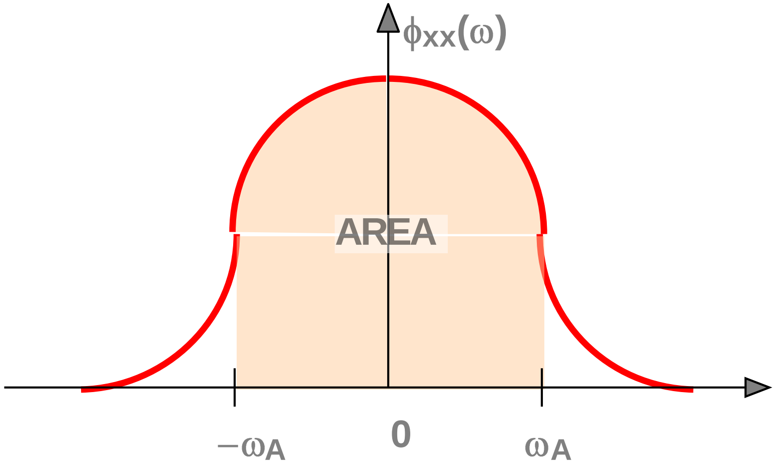
- We can recover the autocorrelation by the inverse Fourier transform

$$\psi_{xx}(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{xx}(\omega) e^{j\omega\tau} d\omega$$

- Note that

$$\phi_{xx}(0) = \int_{-\infty}^{\infty} \psi_{xx}(\tau) d\tau$$

$$\phi_{xx}(\omega) = \phi_{xx}(-\omega)$$



AREA=POWER OF x(t) IN FREQUENCY RANGE, $-\omega_A < \omega < \omega_A$

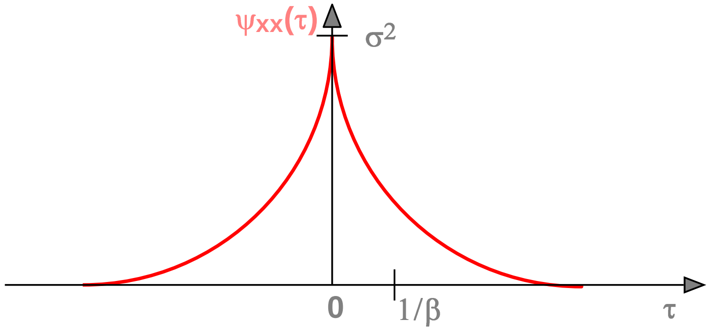
First-Order Markov Process

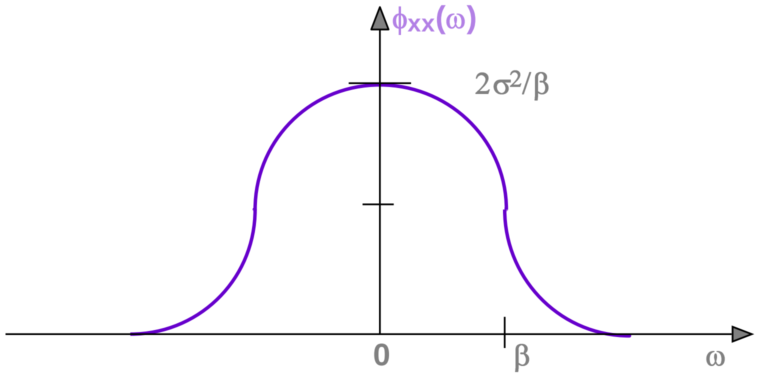
- Simplest example of a random process

$$\psi_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|} \ ; \ \beta > 0$$

$$\phi_{xx}(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2}$$

- $\frac{1}{\beta} =$ correlation time-constant
- $\beta =$ bandwidth of PSD





- Note:

$$\int_{-\infty}^{+\infty} \underbrace{\sigma^2 e^{-\beta|\tau|}}_{\psi_{xx}(\tau)} \cdot e^{-j\omega\tau} d\tau = \frac{2\beta\sigma^2}{\omega^2 + \beta^2} = \phi_{xx}(\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\frac{2\beta\sigma^2}{\omega^2 + \beta^2}}_{\phi_{xx}(\omega)} \cdot e^{j\omega\tau} d\omega = \sigma^2 e^{-\beta|\tau|} = \psi_{xx}(\tau)$$

The Ergodic Hypothesis

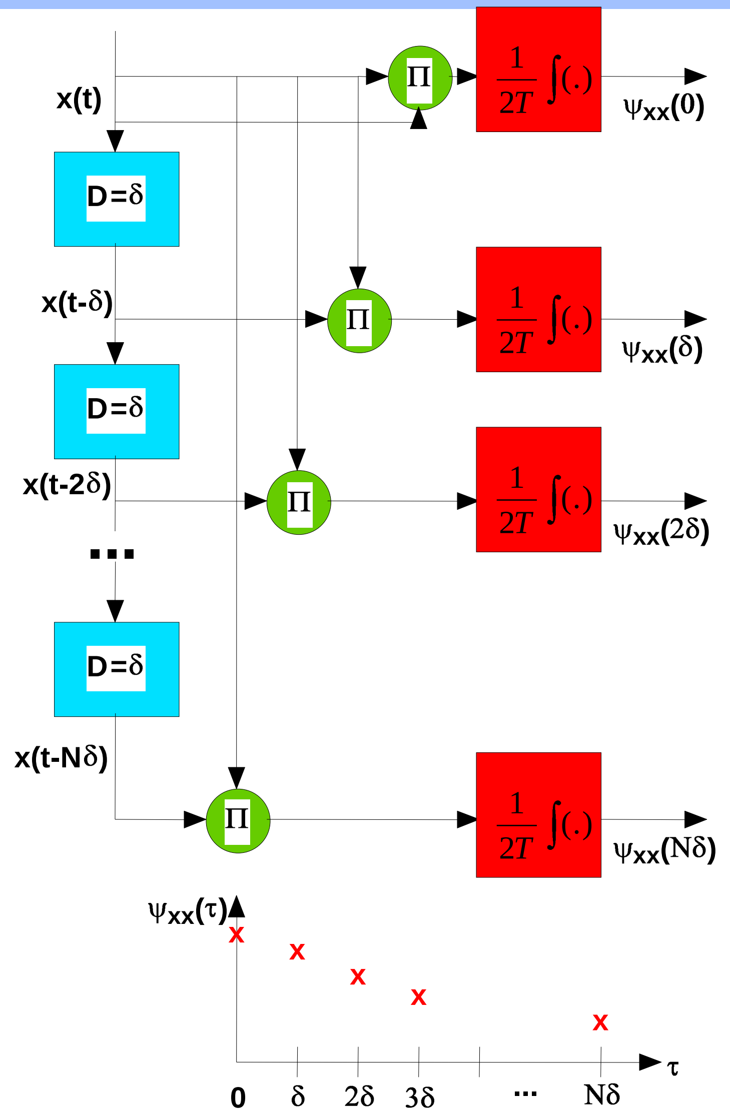
- A stationary random process is *ergodic* if we can calculate its statistics by averaging over time a single "representative" outcome (time function)
- "Representative" means that the time function must reflect all the attributes of the random process (wiggles etc)
- The set of constant random functions is not ergodic, since no outcome is representative

- Mean calculation: $m \equiv E\{x(t)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$
- Variance: $\sigma_{xx}{}^2 \equiv E\{(x(t)-m)^2\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (x(t)-m)^2 dt$
- Autocorrelation function (with $m = 0$):

$$\psi_{xx}(\tau) \equiv E\{x(t)x(\tau)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

Calculating Autocorrelation Functions Using Tapped-Delay Lines

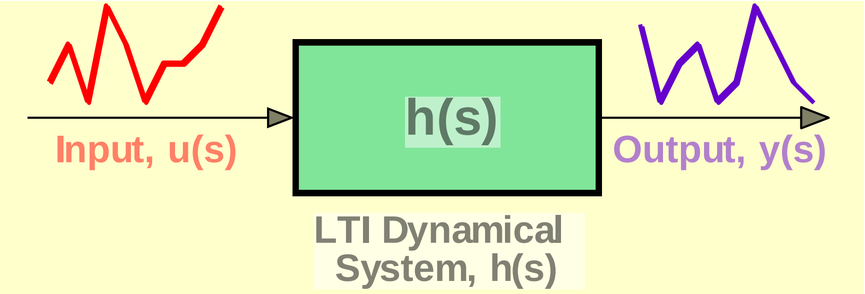
- The autocorrelation function can be approximated by using a tapped-delay line
- Then, the power spectral density (PSD) function can be approximated using discrete Fourier transforms (DFT)



Stochastic SISO LTI Systems

- Given LTI SISO system

  $y(s) = h(s)u(s)$
- Assume $h(s)$ strictly stable
- Random process inputs will generate a random process output
- Want statistical characterization of output random process at steady-state



PROBLEM

- Given PSD $\phi_{uu}(\omega)$ of input $u(t)$
- Find PSD $\phi_{yy}(\omega)$ of output $y(t)$

SOLUTION

$$\phi_{yy}(\omega) = h(j\omega) \cdot h(-j\omega)\phi_{uu}(\omega)$$

$$\Rightarrow$$

$$\phi_{yy}(\omega) = | \; h(j\omega) \; |^2 \cdot \phi_{uu}(\omega)$$

Important Remark

- It is very easy to analyze stochastic LTI systems in the frequency domain
- Very simple algebraic relations linking
  - the PSD of the input random process signal
  - the magnitude of the of the SISO LTI system transfer function as a function of frequency

  to the PSD of the (steady-state) output random process
- We can recover statistical time-domain properties (variance, autocorrelation function) of the output random process by the inverse Fourier transform of the output PSD

Continuous-Time White Noise (WN)

DEFINITION

- Zero-mean, unit intensity white noise, $\xi(t)$

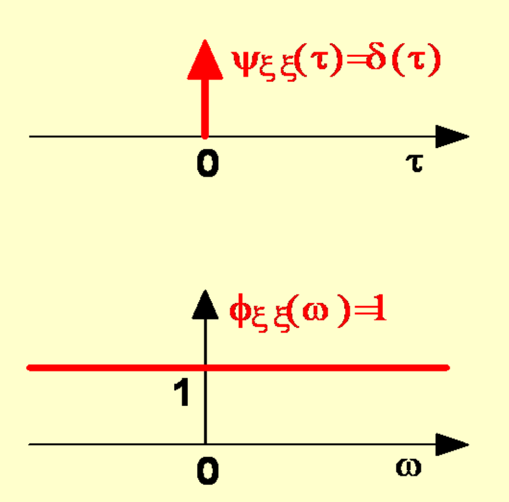
$$E\{\xi(t)\} = 0, \quad cov[\xi(t)\xi(\tau)] \equiv E\{\xi(t)\xi(\tau)\} = \delta(t-\tau)$$

- Autocorrelation function of WN is unit impulse

$$E\{\xi(t)\xi(t+\tau)\} \equiv \psi_{\xi\xi}(\tau) = \delta(\tau)$$

- PSD function of WN is constant for all ω

$$\phi_{\xi\xi}(\omega) = 1 \quad \forall \omega$$



- Continuous-time WN is physical fiction; it is completely unpredictable
  - WN has infinite variance
  - WN has zero time-correlation
  - WN has infinite power
- But, very useful in modeling

WN as Limit of 1st-order Markov Process

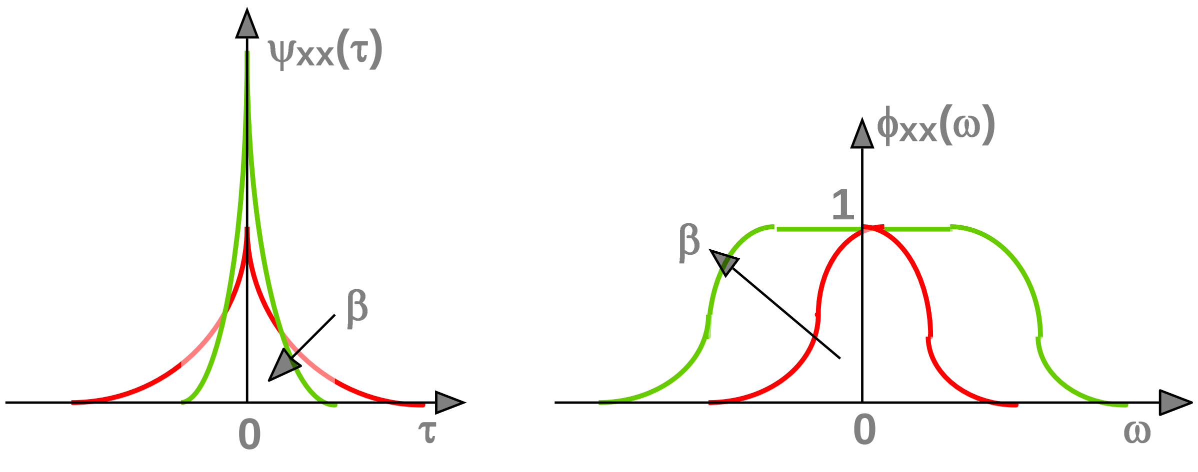
- We can model WN as the limit of a 1st-order Markov process with decreasing correlation time-constant, $\frac{1}{\beta}$
- Consider the 1st-order Markov random process, $x(t)$, with autocorrelation function

$$\psi_{xx}(\tau) = \frac{\beta}{2} e^{-\beta|\tau|} \quad \text{Note that: } \int_{-\infty}^{\infty} \frac{\beta}{2} e^{-\beta|\tau|} d\tau = 1 \quad \forall \beta > 0$$

and associated power spectral density

$$\phi_{xx}(\omega) = \frac{\beta^2}{\omega^2 + \beta^2}$$

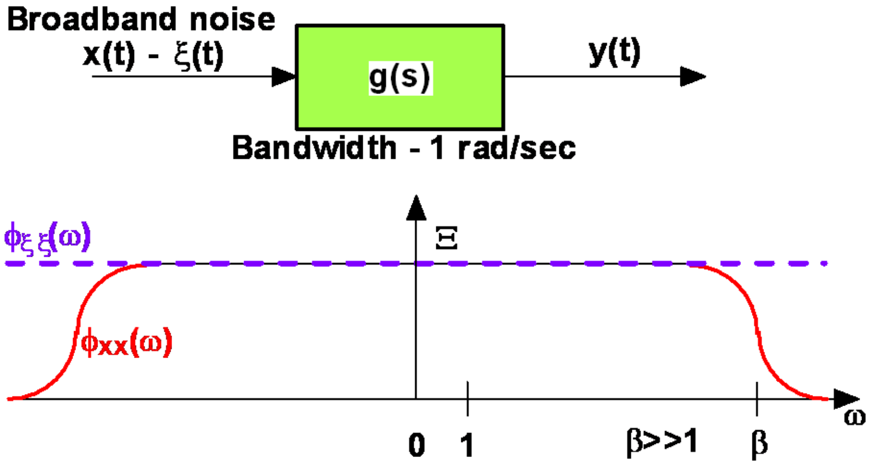
- Then, the unit intensity white noise $\xi(t)$ is the limiting process as $\beta \to \infty$



Comments on White Noise

- White noise can approximate a "broadband" noise process, with constant power density over a wide frequency-range, and which eventually "rolls-off" at very high frequencies
  - we avoid complex models at these high-frequencies
- Continuous white noise is the "most" unpredictable continuous random process, because of its infinite variance and zero time-correlation
  - one can neither estimate nor predict white noise, even though it has been observed for ever
- Pure continuous-time white noise does not exist in nature
  - remember, it has infinite power
- Also, continuous-time white noise is not an "ordinary" mathematical function, so it is easy to make mistakes using white noise in non-rigorous mathematical proofs
  - it belongs to the so-called class of "distribution functions"
  - nevertheless, it is a very useful modeling tool

White Noise Can Approximate Broadband Noise



- Broadband noise RP $x(t)$ has large bandwidth, β, much larger than the bandwidth of the LTI system $g(s)$
- Can approximate output RP PSD, $\phi_{yy}(\omega)$, assuming that input RP $x(t)$ is white noise

- Exact calculation:

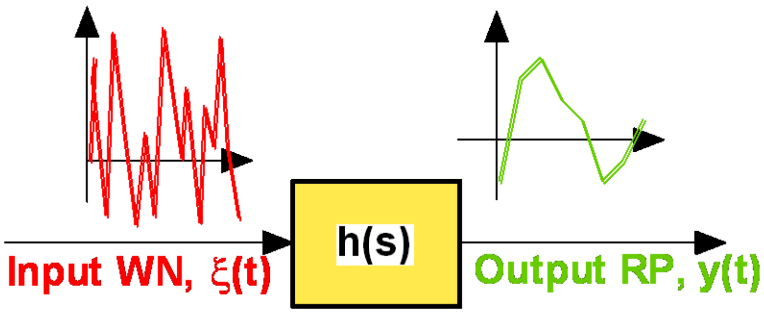
$$\phi_{yy}(\omega) = |\, g(j\omega)\,|^2 \; \phi_{xx}(\omega)$$

- Approximate calculation:

$$\phi_{yy}(\omega) \cong |\, g(j\omega)\,|^2 \cdot \Xi$$

valid for $\beta >> 1$

Prewhitening



- Input WN, $\xi(t)$

$$\psi_{\xi\xi}(\tau) = \delta(\tau); \quad \phi_{\xi\xi}(\omega) = 1$$

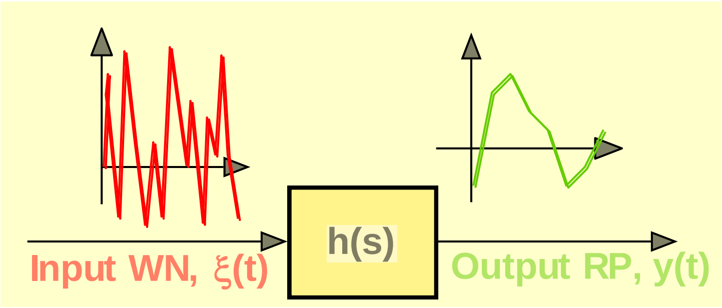
- Output RP $y(t)$ has PSD given by

$$\phi_{yy}(\omega) = h(j\omega)h(-j\omega) = |h(j\omega)|^2 . \underbrace{1}_{\phi_{\xi\xi}(\omega)}$$

- We can always model a physical (colored) stationary random process *y(t)* as the output of a fictitious LTI SISO dynamical system, with transfer function *h(s),* driven by a fictitious white noise input, *ξ(t)*
- This modeling concept is called “prewhitening”

Modeling Using Prewhitening

- Assume that $y(t)$ is ergodic RP
- Measure (experimentally) approximate autocorrelation function, $\psi_{yy}(\tau)$
- Take inverse Fourier transform of $\psi_{yy}(\tau)$ and determine approximation to the PSD of $y(t)$, $\phi_{yy}(\omega)$

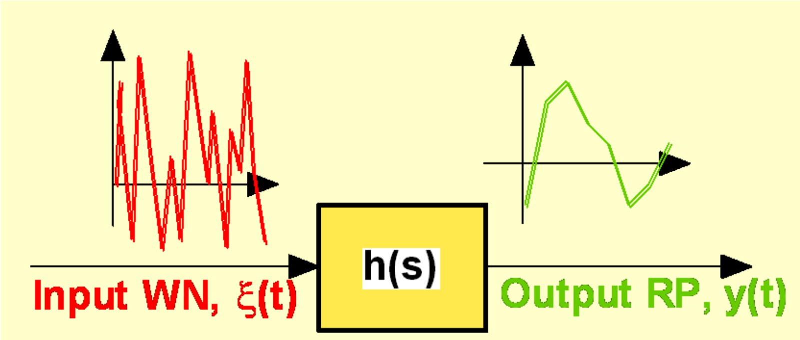


- Find a stable and minimum -phase transfer function, $h(s)$, such that its squared frequency-response $h(j\omega)$ approximates the PSD, i.e.

$$| h(j\omega) |^2 \cong \phi_{yy}(\omega)$$

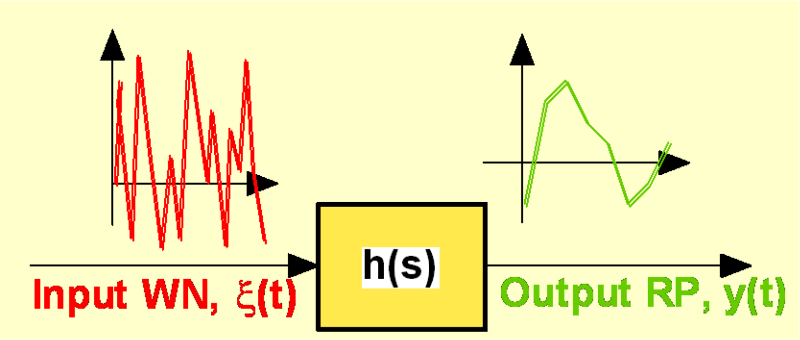
- Determine, if required, a state-space representation for the transfer function $h(s)$
- Think of $y(t)$ as the output of the fictitious $h(s)$ driven by the (also fictitious) unit intensity white noise $\xi(t)$

First-Order Example



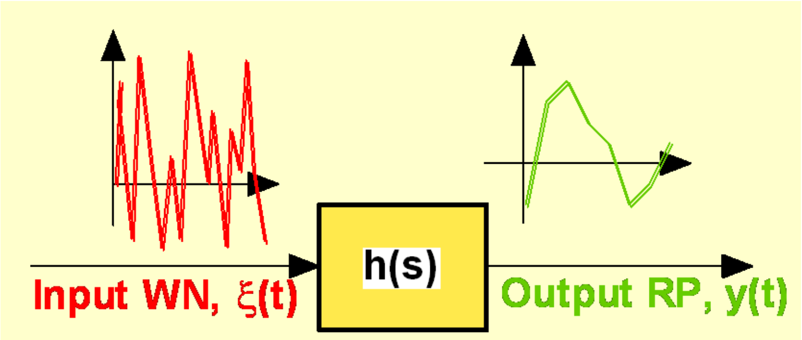
- Variance: $E\left\{y^2(t)\right\} = \sigma^2$
- Autocorrelation function: $\psi_{yy}(\tau) = \sigma^2 e^{-\beta|\tau|}$
- Power spectral density: $\phi_{yy}(\omega) = \dfrac{2\beta\sigma^2}{\omega^2 + \beta^2}$
- Transfer Function: $h(s) = \dfrac{\sigma\sqrt{2\beta}}{s + \beta}$
- Ref. [1], p.44

A Second-Order Example



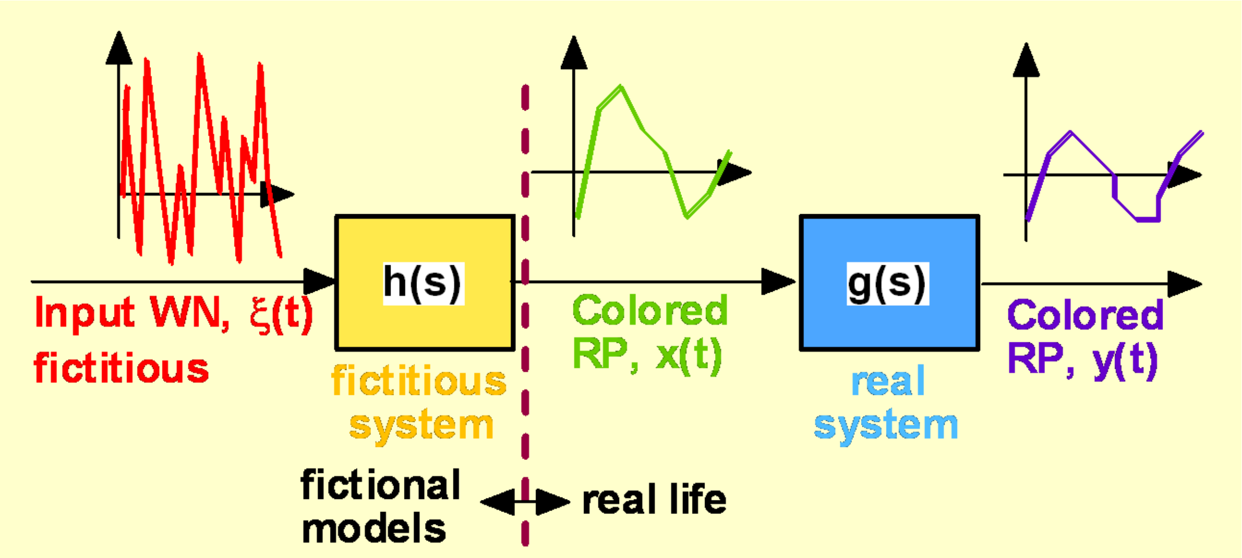
- Variance: $E\left\{y^2(t)\right\} = \sigma^2$
- Autocorrelation function: $\psi_{yy}(\tau) = \sigma^2 e^{-\beta|\tau|}\left(1 + \beta\,|\,\tau\,|\right)$
- Power spectral density: $\phi_{yy}(\omega) = \dfrac{4\beta^3\sigma^2}{\left(\omega^2 + \beta^2\right)^2}$
- Transfer function: $h(s) = \dfrac{2\sigma\beta^{3/2}}{s^2 + \sqrt{2}\beta s + \beta^2}$
- Ref. [1], p.44

Another Second-Order Example



- Variance: $E\left\{y^2(t)\right\} = \sigma^2$
- Autocorrelation function: $\psi_{yy}(\tau) = \dfrac{\sigma^2}{\cos\theta} e^{-\varsigma\omega_n|\tau|} \cos\left(\sqrt{1-\varsigma^2}\,\omega_n |\tau| - \theta\right)$
- Power spectral density: $\phi_{yy}(\omega) = \sigma^2 \cdot \dfrac{a^2\omega^2 + b^2}{\omega^4 + 2\omega_n^2(2\varsigma^2 - 1)\omega^2 + \omega_n^4}$
- Transfer function: $h(s) = \sigma \cdot \dfrac{as + b}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$
- Ref. [8], p. 72

Modeling Implications



- The output random process, *y(t),* of a “real” system *g(s)* to a colored input random process, *x(t),* can also be modeled by the cascaded system *g(s)h(s),* where *h(s)* is the “prewhitening” system for the random process *x(t)*

Remarks

- Continuous-time random processes are essential in modeling the impact of random disturbances and “noise” on physical systems
- It is crucial to appreciate, and fully understand, the time-domain and frequency-domain properties of stationary random processes, via the associated autocorrelation and PSD function
- The power spectral density of stationary random processes is a very powerful tool when we analyze the input and output signals, of a SISO LTI system, as random processes
- Even though a physical fiction, continuous-time white noise is a powerful modeling tool
- All SISO results will be extended to the multi-input multi-output (MIMO) case, fully taking advantage of state-space representations

Vector Random Processes (VRPs)

- All definitions and results for the scalar case readily extend to the case of vector-valued random processes
- A VRP $x(t) \in R^n$ is a n-dimensional column vector

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots\cdot \\ x_n(t) \end{bmatrix}$$

whose elements, $x_i(t)$, are scalar-valued random processes

PDF and Mean For Nonstationary VRP

- All elements $x_i(t);\ i = 1, 2, \ldots, n,$ are jointly - distributed RPs
- In the nonstationary case the pdf of the VRP is the

  scalar - valued function $p(x(t), t) = p(x_1(t), x_2(t), \ldots, x_n(t), t)$

  with mean $\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \ldots \\ \bar{x}_n(t) \end{bmatrix} \equiv E\{x(t)\} = \int x(t)\, p(x(t), t)\, dx(t)$

  which is shorthand for

$$\bar{x}_i(t) = E\{x_i(t)\} =$$

$$= \int\int \ldots \int x_i(t)\, p(x_1(t), x_2(t), \ldots, x_n(t), t)\, dx_1(t)\, dx_2(t) \ldots dx_n(t)$$

Covariance Matrix For Nonstationary VRP

- The nxn covariance matrix of the nonstationary vector random process $x(t) \in R^n$ is defined by

$$\Sigma(t) = cov[x(t); x(t)] \equiv E\left\{(x(t) - \bar{x}(t))(x(t) - \bar{x}(t))'\right\} =$$

$$= \int (x(t) - \bar{x}(t))(x(t) - \bar{x}(t))' \, p(x(t), t) dx(t)$$

- The nxn covariance matrix is symmetric and positive - semidefinite

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) & \dots & \Sigma_{1n}(t) \\ \Sigma_{12}(t) & \Sigma_{22}(t) & \dots & \Sigma_{2n}(t) \\ \dots & \dots & \dots & \dots \\ \Sigma_{1n}(t) & \Sigma_{2n}(t) & \dots & \Sigma_{nn}(t) \end{bmatrix}; \quad \Sigma(t) = \Sigma'(t) \geq 0$$

where, element - by - element,

$$\Sigma_{ij}(t) = E\left\{(x_i(t) - \bar{x}_i(t))(x_j(t) - \bar{x}_j(t))\right\} =$$

$$= \int \int \dots \int (x_i(t) - \bar{x}_i(t))(x_j(t) - \bar{x}_j(t)) p(x_1(t), \dots, x_n(t), t) dx_1(t) \dots dx_n(t)$$

PDF and Mean For Stationary VRP

- All elements $x_i(t);\ i = 1, 2, \dots, n,$ are jointly - distributed RPs
- In the stationary case the pdf of the VRP is the

  scalar - valued function, $p(x(t)) = p\big(x_1(t), x_2(t), \dots, x_n(t)\big),$

  which does not depend explicitly on time,

  with mean $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \dots \\ \bar{x}_n \end{bmatrix} \equiv E\{x(t)\} = \int x(t)\, p\big(x(t), t\big) dx(t) =$ constant

  which is shorthand for

  $$\bar{x}_i = E\{x_i(t)\} =$$

  $$= \int \int \dots \int x_i(t)\, p\big(x_1(t), x_2(t), \dots, x_n(t)\big) dx_1(t) dx_2(t) \dots dx_n(t)$$

Covariance Matrix For Stationary VRP

- The nxn covariance matrix of the stationary vector random process $x(t) \in R^n$ is constant and is defined by

$$\Sigma = cov[x(t);\, x(t)] \equiv E\left\{ (x(t) - \bar{x}(t))(x(t) - \bar{x}(t))' \right\} =$$

$$= \int (x(t) - \bar{x}(t))(x(t) - \bar{x}(t))' \, p(x(t))\, dx(t)$$

- The nxn covariance matrix is symmetric and positive - semidefinite

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1n} \\ \Sigma_{12} & \Sigma_{22} & \dots & \Sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \Sigma_{1n} & \Sigma_{2n} & \dots & \Sigma_{nn} \end{bmatrix}; \quad \Sigma = \Sigma' \geq 0$$

where, element - by - element,

$$\Sigma_{ij} = E\left\{ \left(x_i(t) - \bar{x}_i(t)\right)\left(x_j(t) - \bar{x}_j(t)\right) \right\} =$$

$$= \int \int \dots \int \left(x_i(t) - \bar{x}_i(t)\right)\left(x_j(t) - \bar{x}_j(t)\right) p\left(x_1(t), \dots, x_n(t)\right) dx_1(t) \dots dx_n(t)$$

Correlation and PSD Matrices

- For stationary zero-mean vector random processes, the correlation matrix is defined by

  $$\Psi_{xx}(\tau) \equiv E\{x(t)x'(t+\tau)\}$$

  with elements $\psi_{x_i x_j}(\tau) \equiv E\{x_i(t)x_j(t+\tau)\}$
- The PSD matrix is denoted by $\Phi_{xx}(\omega)$, whose elements are computed by the Fourier transform of the associated correlation function

  $$\phi_{x_i x_j}(\omega) = \int_{-\infty}^{\infty} \psi_{x_i x_j}(\tau) e^{-j\omega\tau} d\tau$$
- Formally,

  $$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \Psi_{xx}(\tau) e^{-j\omega\tau} d\tau; \quad \Psi_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega) e^{j\omega\tau} d\omega$$

Vector White Noise

- Nonstationary case: $\xi(t) \in R^m$ is vector white noise, with

  $$E\{\xi(t)\} = 0, \quad cov[\xi(t);\, \xi(\tau)] \equiv E\{\xi(t)\xi'(\tau)\} = \Xi(t) \cdot \delta(t-\tau)$$

- Stationary case: $\xi(t) \in R^m$ is vector white noise, with

  $$E\{\xi(t)\} = 0, \quad cov[\xi(t);\, \xi(\tau)] \equiv E\{\xi(t)\xi'(\tau)\} = \Xi \cdot \delta(t-\tau)$$

  and correlation matrix $\Psi_{\xi\xi}(\tau) = \Xi \cdot \delta(\tau)$

  and power spectral density matrix $\Phi_{\xi\xi}(\omega) = \Xi$

- In either case, we refer to $\Xi(t)$ or Ξ as the "intensity matrix"
- By the law of large numbers, white noise is gaussian

Gaussian Vector Random Processes

- In the nonstationary case, $x(t) \in R^n$, the gaussian PDF takes the form

$$p(x(t), t) = \frac{1}{(2\pi)^{n/2}\sqrt{det\, \Sigma(t)}} \cdot exp\left\{ -\frac{1}{2}\left(x(t) - \bar{x}(t)\right)' \Sigma^{-1}(t)\left(x(t) - \bar{x}(t)\right) \right\}$$

- Often, we use the abbreviation $x(t) \sim N(\bar{x}(t), \Sigma(t))$
- In the stationary case, $x(t) \in R^n$, the mean and covariance are constant so that the gaussian PDF takes the form

$$p(x(t)) = \frac{1}{(2\pi)^{n/2}\sqrt{det\, \Sigma}} \cdot exp\left\{ -\frac{1}{2}\left(x(t) - \bar{x}\right)' \Sigma^{-1}\left(x(t) - \bar{x}\right) \right\}$$

- Often, we use the abbreviation $x(t) \sim N(\bar{x}, \Sigma)$

Remarks on Vector Random Processes

- We postpone till later the topic of how vector random processes interact with linear dynamic systems
- Such manipulations will require extensive use of state-space methods and models

References

[1]. A. Gelb, *Applied Optimal Estimation*, MIT Press, 1974, Chapter 2

[2]. A. Papoulis, *Probability, Random Variables and Stochastic Processes*, McGraw Hill, 1988

[3]. W.B. Davenport, Jr. and W.L. Root, *An Introduction to Random Signals and Noise*, McGraw Hill, 1958

[4]. J.H. Laning, Jr. and R.H. Battin, *Random Processes in Automatic Control*, McGraw Hill, 1956

[5]. F.C. Schweppe, *Uncertain Dynamic Systems*, Prentice Hall, 1973

[6]. A.H. Jazwinsky, *Stochastic Processes and Filtering Theory*, Academic Press, 1970

[7]. T. Kailath, *Lectures on Wiener and Kalman Filtering*, Springer-Verlag, 1981

[8]. M.S. Grewal and A.P. Andrews, *Kalman Filtering:Theory and Practice*, Prentice-Hall, 1993