REF: KF #2

1

STOCHASTIC ESTIMATION

Review of Probabilistic Concepts

2

SCALAR RANDOM VARIABLES

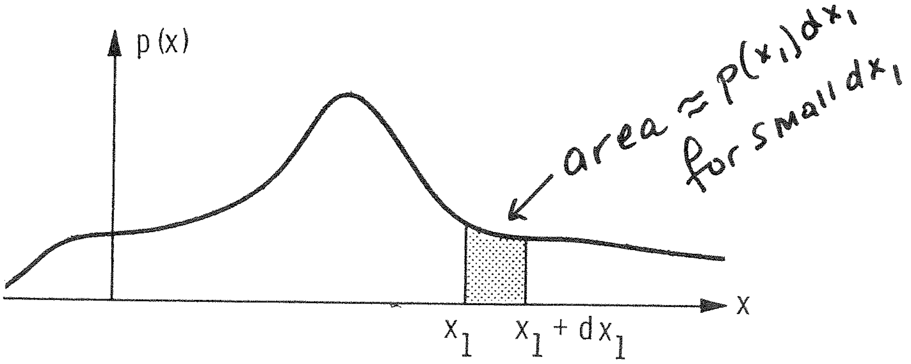
- Let x be a random variable, the outcome of a probabilistic experiment.
- Let $p(x)$ denote probability density function of x.

Then

$$\boxed{p(x_1)\,dx_1 = \text{probability that the value of } x \text{ will be in the interval } x_1 \le x \le x_1 + dx_1} \qquad (1)$$

3

VISUALIZATION



Note:

$$p(x) \ge 0 \;; \quad \int_{-\infty}^{+\infty} p(x)\,dx = 1 \qquad (2)$$

4
EXPECTED VALUES

- Definition

$$E\{x\} \triangleq \bar{x} \triangleq \int_{-\infty}^{+\infty} x\, p(x)\, dx \qquad (3)$$

$$E\{f(x)\} = \int_{-\infty}^{+\infty} f(x)\, p(x)\, dx \qquad (4)$$

- Note:

$$E\{x-\bar{x}\} = 0 \qquad (5)$$

from (4) with $f(x) = x - \bar{x}$

5
VARIANCE

$$\mathrm{var}[x] \triangleq E\{(x-\bar{x})^2\} \qquad (6)$$

from (4) with $f(x) = (x-\bar{x})^2$

$$\triangleq \int_{-\infty}^{+\infty} (x-\bar{x})^2 p(x)\, dx$$

- variance = (standard deviation)^2
- We shall also write

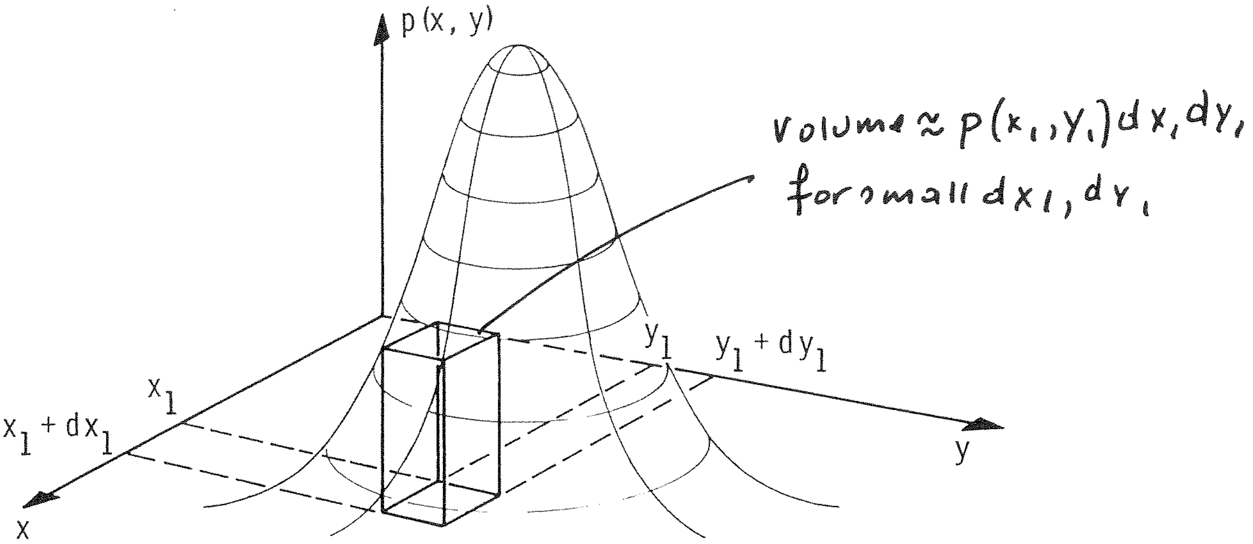
$$\mathrm{cov}[x; x] = \mathrm{var}[x] \qquad (7)$$

6
JOINT DENSITIES

- x and y are random variables.
- p(x, y) = joint probability density function

$$\boxed{\begin{array}{l} p(x_1, y_1)\, dx_1\, dy_1 = \text{probability that} \\ \qquad x_1 \le x \le x_1 + dx_1 \quad \underline{\text{AND}} \\ \qquad y_1 \le y \le y_1 + dy_1 \end{array}} \qquad (8)$$

7



8

EXPECTED VALUES

$$\bar{x} \triangleq E\left\{x\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, p(x, y)\, dx\, dy \qquad (9)$$

$$\bar{y} \triangleq E\left\{y\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, p(x, y)\, dx\, dy \qquad (10)$$

$$E\left\{f(x, y)\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\, p(x, y)\, dx\, dy \qquad (11)$$

9

VARIANCES

$$\mathrm{var}\left[x\right] \triangleq \Sigma_x \triangleq E\left\{(x-\bar{x})^2\right\} \qquad (12)$$

from (11) using $f(x,y) = (x-\bar{x})^2$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\bar{x})^2\, p(x, y)\, dx\, dy$$

$$\mathrm{var}\left[y\right] = \Sigma_y = E\left\{(y-\bar{y})^2\right\} \qquad (13)$$

from (11) with $f(x,y) = (y-\bar{y})^2$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (y-\bar{y})^2\, p(x, y)\, dx\, dy$$

10
COVARIANCE AND CORRELATION

- Covariance

$$\Sigma_{xy} \triangleq \mathrm{cov}\left[x;y\right] \triangleq E\left\{(x-\bar{x})(y-\bar{y})\right\} \quad (14)$$

$$\triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\bar{x})(y-\bar{y})\,p(x,y)\,dx\,dy$$

$$= \mathrm{cov}\left[y;x\right]$$

from (11) with
$f(x,y) = (x-\bar{x})(y-\bar{y})$

- Correlation (common in statistical analyses)

$$\rho(x,y) = \frac{\Sigma_{xy}}{\sqrt{\Sigma_x \Sigma_y}} \quad (15)$$

$$\left|\rho(x,y)\right| \le 1 \quad (16)$$

11
INDEPENDENCE

- If x and y are independent random variables

$$p(x,y) = p(x)\,p(y) \quad (17)$$

- Implications

$$E\left\{xy\right\} = E\left\{x\right\} E\left\{y\right\} = \bar{x}\,\bar{y} \quad (18)$$

$$\mathrm{cov}\left[x;y\right] = E\left\{(x-\bar{x})(y-\bar{y})\right\} \quad (19)$$

$$= E\left\{x-\bar{x}\right\} \cdot E\left\{y-\bar{y}\right\} = 0$$

$$\rho(x,y) = 0 \quad (20)$$

If $\rho(x,y) = 0$, this does not necessarily mean that x and y are independent RVs.

12
PROPERTIES

$$p(x,y) \ge 0 \quad (21)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x,y)\,dx\,dy = 1 \quad (22)$$

13
OTHER DEFINITIONS

- Marginal density

$$p(x) = \int_{-\infty}^{+\infty} p(x, y)\, dy \tag{23}$$

$$p(y) = \int_{-\infty}^{+\infty} p(x, y)\, dx \tag{24}$$

- Conditional density

$$p(x \mid y) \quad \text{or} \quad p(y \mid x) \tag{25}$$

14
INTERPRETATION

$p(x, y)$ = joint prob. density function (PDF) (26)

$p(x \mid y)$ = conditional prob. density function of x given the value of y (27)

$p(x_1 \mid y_1)\, dx_1$	= probability that $x_1 \le x \le x_1 + dx_1$ given that we have seen the value $y = y_1$

(28)

15
BAYES RULE

Relates joint, marginal, and conditional PDFs

$$\boxed{\begin{aligned} p(x, y) &= p(x \mid y)\, p(y) \\ &= p(y \mid x)\, p(x) \end{aligned}} \tag{29}$$

- If x, y independent

$$p(x \mid y) = p(x) \tag{30}$$

$$p(y \mid x) = p(y) \tag{31}$$

16
CONDITIONAL EXPECTATIONS

$$E\left\{x \mid y\right\} = \int_{-\infty}^{+\infty} x\, p(x \mid y)\, dx \qquad (32)$$

$$= \text{a function of } y$$

$$E\left\{y \mid x\right\} = \int_{-\infty}^{+\infty} y\, p(y \mid x)\, dy \qquad (33)$$

$$= \text{a function of } x$$

- Note: $E\left\{x \mid y\right\} \neq E\left\{x\right\}$
  $E\left\{y \mid x\right\} \neq E\left\{y\right\}$ equality would hold if x and y were independent RVs

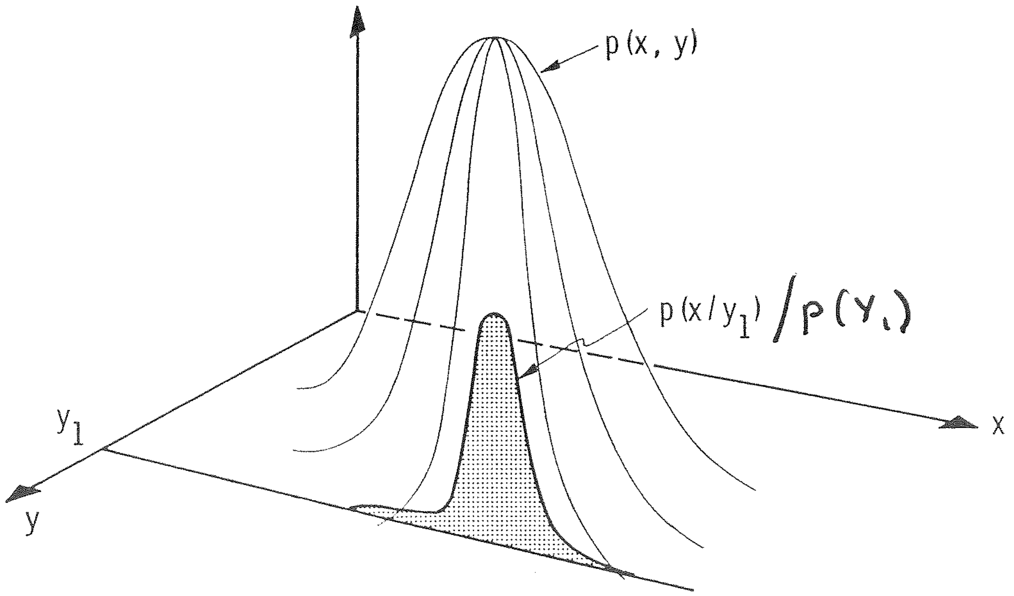
17
CONDITIONAL VARIANCES

$$\mathrm{var}\left[x \mid y\right] = E\left\{\left(x - E\left\{x \mid y\right\}\right)^2 \,\middle|\, y\right\} \qquad (34)$$

$$= \int_{-\infty}^{+\infty} \left(x - E\left\{x \mid y\right\}\right)^2 p(x \mid y)\, dx$$

$$= \text{a function of } y$$

18



19

Example

- x = weight
  y = height

$E\{x\} = 152$ lbs
$E\{y\} = 5$ ft 8 in

$E\{x \mid y = 6 \text{ ft } 3 \text{ in}\} = 195$ lbs
$E\{y \mid x = 30 \text{ lbs}\} = 3$ ft 2 in

20

RANDOM VECTORS

- $\underline{x}$ n-dimensional random vector

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \qquad (35)$$

$x_1, x_2, \ldots, x_n$ are scalar real random variables, jointly distributed

All x_i may be dependent RVs!

- Joint Density (joint PDF)

$$p(\underline{x}) = p\left(x_1, x_2, \ldots, x_n\right) \qquad (36)$$

21

EXPECTATION

$$E\{\underline{x}\} \triangleq \underline{\bar{x}} \triangleq \int_{-\infty}^{+\infty} \underline{x}\, p(\underline{x})\, d\underline{x} \qquad (37)$$

Shorthand for (39)

$$E\{\underline{x}\} = \begin{bmatrix} E\{x_1\} \\ E\{x_2\} \\ \cdots \\ E\{x_n\} \end{bmatrix} \qquad (38)$$

$$E\{x_i\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} x_i\, p(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \qquad (39)$$

$i = 1, 2, \ldots, n$

$$E\{\underline{f}(\underline{x})\} = \int_{-\infty}^{+\infty} \underline{f}(\underline{x})\, p(\underline{x})\, d\underline{x} \qquad (40)$$

22

Example: $\underline{A}$, $\underline{b}$ deterministic use $f(x) = \underline{A}\,\underline{x} + \underline{b}$ in (40)

$$E\left\{\underline{A}\,\underline{x} + \underline{b}\right\} = \int_{-\infty}^{+\infty} (\underline{A}\,\underline{x} + \underline{b})\, p(\underline{x})\, \underline{dx} \tag{41}$$

$$= \underline{A} \underbrace{\int_{-\infty}^{+\infty} \underline{x}\, p(\underline{x})\, \underline{dx}}_{E\{\underline{x}\}} + \underline{b} \underbrace{\int_{-\infty}^{+\infty} p(\underline{x})\, \underline{dx}}_{1}$$

$$\longrightarrow E\left\{\underline{A}\,\underline{x} + \underline{b}\right\} = \underline{A}\, E\left\{\underline{x}\right\} + \underline{b} \tag{42}$$

23

COVARIANCE MATRIX

- $\underline{x} \in R_n$, random vector

$$\underline{\Sigma} \triangleq \mathrm{cov}\left[\underline{x};\underline{x}\right] \triangleq E\left\{(\underline{x}-\underline{\bar{x}})(\underline{x}-\underline{\bar{x}})'\right\} \tag{43}$$

$$\underline{\Sigma} \triangleq \begin{bmatrix} \mathrm{cov}[x_1;x_1] & \mathrm{cov}[x_1;x_2] & \cdots & \mathrm{cov}[x_1;x_n] \\ \mathrm{cov}[x_1;x_2] & \mathrm{cov}[x_2;x_2] & \cdots & \mathrm{cov}[x_2;x_n] \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{cov}[x_1;x_n] & \mathrm{cov}[x_2;x_n] & \cdots & \mathrm{cov}[x_n;x_n] \end{bmatrix} \tag{44}$$

- $\underline{\Sigma}$ symmetric, positive semidefinite $n \times n$ matrix.

24

LINEAR COMBINATIONS OF INDEPENDENT RANDOM VECTORS

- $\underline{x}$, $\underline{y}$ independent random vectors $\Rightarrow$

$$p(\underline{x}, \underline{y}) = p(\underline{x})\, p(\underline{y}) \tag{45}$$

- Mean Calculation

$$E\left\{\underline{A}\,\underline{x} + \underline{B}\,\underline{y}\right\} = \underline{A}\, E\left\{\underline{x}\right\} + \underline{B}\, E\left\{\underline{y}\right\} \tag{46}$$

True also for dependent RVs

- Note:

$$\mathrm{cov}\left[\underline{x};\underline{y}\right] = \underline{0} \tag{47}$$

25

- Covariance Calculation ($\underline{x}$ and $\underline{y}$ independent)

$$\underline{z} = \underline{A}\,\underline{x} + \underline{B}\,\underline{y} \tag{48}$$

- Let

$$\underline{\Sigma}_z = \text{cov}\left[\underline{z};\underline{z}\right] \tag{49}$$

$$\left.\begin{aligned}\underline{\Sigma}_x &= \text{cov}\left[\underline{x};\underline{x}\right]\\ \underline{\Sigma}_y &= \text{cov}\left[\underline{y};\underline{y}\right]\end{aligned}\right\} \tag{50}$$

- Then

$$\boxed{\underline{\Sigma}_z = \underline{A}\,\underline{\Sigma}_x\,\underline{A}' + \underline{B}\,\underline{\Sigma}_y\,\underline{B}'} \tag{51}$$

Key equation!

26

GAUSSIAN RANDOM VARIABLES

(Gaussian = normal)

- x: scalar Gaussian random variable

$$E\{x\} = \bar{x}\,, \qquad \text{var}\left[x\right] = \Sigma \tag{52}$$

- Gaussian Density Function

$$p(x) = \frac{1}{\sqrt{2\pi}\sqrt{\Sigma}}\exp\left\{-\frac{1}{2\Sigma}(x-\bar{x})^2\right\} \tag{53}$$

often we use shorthand

$x \sim N(\bar{x}, \Sigma)$

27

GAUSSIAN RANDOM VECTORS

- $\underline{x}$: n-dimensional Gaussian random vector with

$$\underline{\bar{x}} = E\{\underline{x}\}\,; \quad \underline{\Sigma} = \text{cov}\left[\underline{x};\underline{x}\right] \tag{54}$$

often we use shorthand

$\underline{x} \sim N(\underline{\bar{x}}, \underline{\Sigma})$

- Gaussian Density Function

$$p(\underline{x}) = (2\pi)^{-\frac{n}{2}}\,(\det\underline{\Sigma})^{-\frac{1}{2}} \tag{55}$$

$$\exp\left\{-\frac{1}{2}(\underline{x}-\underline{\bar{x}})'\,\underline{\Sigma}^{-1}\,(\underline{x}-\underline{\bar{x}})\right\}$$

28

FACTS

- If $\underline{x}$, $\underline{y}$ independent Gaussian random vectors
- Then, the random vector

$$\underline{z} = \underline{A}\,\underline{x} + \underline{B}\,\underline{y} \tag{56}$$

is also Gaussian.

- Rule: Linearity preserves gaussian distribution.

God made gaussian RVs and linear transformations to live in harmony!