REF. NO. KF #3

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STOCHASTIC ESTIMATION

Response of Linear Systems to White Noise Inputs: Discrete Time Case

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MATHEMATICAL DEFINITION OF DISCRETE-TIME WHITE NOISE

- Time index

$$t = 0, 1, 2, \ldots$$

Examples: Dice, Roulette

- Discrete white noise $\underline{\xi}(t)$
- Mean:

$$E\{\underline{\xi}(t)\} = \underline{\bar{\xi}}(t) \qquad (1)$$

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- Covariance:

$$\begin{aligned} &\operatorname{cov}[\underline{\xi}(t); \underline{\xi}(\tau)] \\ &= E\{(\underline{\xi}(t) - \underline{\bar{\xi}}(t))(\underline{\xi}(\tau) - \underline{\bar{\xi}}(\tau))'\} \\ &= \underline{\Xi}(t)\,\delta_{t\tau} \end{aligned} \qquad (2)$$

Finite variance!
Zero time-correlation

- $\delta_{t\tau}$ is Kroenecker delta

$$\delta_{t\tau} = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau \end{cases} \qquad (3)$$

$$\underline{\Xi}(t) = \underline{\Xi}'(t) \geq \underline{0} \quad \text{intensity matrix} \qquad (4)$$

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PROBLEM FORMULATION

- State Dynamics:

Deterministic

$$\boxed{\begin{aligned}\underline{x}(t+1) = \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t)\\ + \underline{L}(t)\,\underline{\xi}(t)\end{aligned}} \qquad (5)$$

discrete-time WN

- Initial State $\underline{x}(0)$: random vector

  $\underline{\xi}(t)$: discrete white noise

- $\underline{A}(t)$, $\underline{B}(t)$, $\underline{L}(t)$, $\underline{u}(t)$: deterministic

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BASIC QUESTIONS

How does $\underline{x}(t)$ propagate on the average?

⟶ Determine $E\left\{\underline{x}(t)\right\} \triangleq \bar{\underline{x}}(t)$

- How variable is $\underline{x}(t)$ about its average?

⟶ Determine $\mathrm{cov}\left[\underline{x}(t); \underline{x}(t)\right] \triangleq \underline{\Sigma}(t)$

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INITIAL STATE UNCERTAINTY

- $\underline{x}(0) \in R_n$ is random vector

$$E\left\{\underline{x}(0)\right\} = \bar{\underline{x}}(0) \qquad (6)$$

$$\mathrm{cov}\left[\underline{x}(0); \underline{x}(0)\right] = \underline{\Sigma}(0) \qquad (7)$$

$$= \underline{\Sigma}'(0) \geq \underline{0}$$

- $\bar{\underline{x}}(0)$, $\underline{\Sigma}(0)$ assumed known

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DRIVING WHITE NOISE

- $\underline{\xi}(t)$ white discrete time noise sequence

$$E\left\{\underline{\xi}(t)\right\} = \bar{\underline{\xi}}(t) \qquad (8)$$

$$\mathrm{cov}\left[\underline{\xi}(t); \underline{\xi}(\tau)\right] = \underline{\Xi}(t)\,\delta_{t\tau} \qquad (9)$$

$$\underline{\Xi}(t) = \underline{\Xi}'(t) \geq \underline{0}$$

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- $\underline{\xi}(t)$, $\underline{x}(0)$ independent for all t

$$\longrightarrow \quad \text{cov}\left[\underline{\xi}(t); \underline{x}(0)\right] = \underline{0} \qquad (10)$$

- $\bar{\underline{\xi}}(t)$, $\underline{\Xi}(t)$ assumed known

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SUMMARY OF RESULTS

- State sequence $\underline{x}(t)$ is Markov sequence.
- Mean Dynamics

$$E\left\{\underline{x}(t)\right\} \triangleq \bar{\underline{x}}(t) \qquad (11)$$

$$\boxed{\begin{aligned}\bar{\underline{x}}(t+1) &= \underline{A}(t)\,\bar{\underline{x}}(t) + \underline{B}(t)\,\underline{u}(t)\\ &+ \underline{L}(t)\,\bar{\underline{\xi}}(t)\end{aligned}} \qquad (12)$$

Dynamic propagation of mean. Pure prediction.

$$\bar{\underline{x}}(0) = E\left\{\underline{x}(0)\right\} = \text{known} \qquad (13)$$

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- Covariance Dynamics

$$\underline{\Sigma}(t) \triangleq \text{cov}\left[\underline{x}(t); \underline{x}(t)\right] \qquad (14)$$

$$= E\left\{\left(\underline{x}(t) - \bar{\underline{x}}(t)\right)\left(\underline{x}(t) - \bar{\underline{x}}(t)\right)'\right\}$$

$$\boxed{\begin{aligned}\underline{\Sigma}(t+1) &= \underline{A}(t)\,\underline{\Sigma}(t)\,\underline{A}'(t)\\ &+ \underline{L}(t)\,\underline{\Xi}(t)\,\underline{L}'(t)\end{aligned}} \qquad (15)$$

Dynamic propagation of covariance (uncertainty).

OFF-LINE

$$\underline{\Sigma}(0) = \text{cov}\left[\underline{x}(0); \underline{x}(0)\right] = \text{known} \qquad (16)$$

No need to know $\underline{u}(t)$.

What-if scenarios.

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DERIVATION OF MEAN DYNAMICS

- State Dynamics

$$\underline{x}(t+1) = \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\underline{\xi}(t) \tag{17}$$

$$E\left\{\underline{x}(t+1)\right\} = E\left\{\underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\underline{\xi}(t)\right\} \tag{18}$$

$$= \underline{A}(t)\,\underbrace{E\left\{\underline{x}(t)\right\}}_{\bar{\underline{x}}(t)} + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\underbrace{E\left\{\underline{\xi}(t)\right\}}_{\bar{\underline{\xi}}(t)}$$

- Therefore

$$\boxed{\bar{\underline{x}}(t+1) = \underline{A}(t)\,\bar{\underline{x}}(t) + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\bar{\underline{\xi}}(t)} \tag{19}$$

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"Error" Dynamics (Prediction error)

$$\tilde{\underline{x}}(t) \triangleq \underline{x}(t) - \bar{\underline{x}}(t)\,; \quad E\left\{\tilde{\underline{x}}(t)\right\} = \underline{0} \tag{20}$$

- From Eqs (17) and (19)

$$\boxed{\tilde{\underline{x}}(t+1) = \underline{A}(t)\,\tilde{\underline{x}}(t) + \underline{L}(t)\,\tilde{\underline{\xi}}(t)} \tag{21}$$

where

$$\tilde{\underline{\xi}}(t) = \underline{\xi}(t) - \bar{\underline{\xi}}(t) \tag{22}$$

$$E\left\{\tilde{\underline{\xi}}(t)\right\} = \underline{0} \tag{23}$$

$$\text{cov}\left[\tilde{\underline{\xi}}(t);\tilde{\underline{\xi}}(\tau)\right] = E\left\{\tilde{\underline{\xi}}(t)\,\tilde{\underline{\xi}}'(\tau)\right\} \tag{24}$$

$$= \underline{\Xi}(t)\,\delta_{t\tau}$$

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DERIVATION OF COVARIANCE DYNAMICS (Error Covariance)

- State covariance matrix $\underline{\Sigma}(t)$

$$\underline{\Sigma}(t) \triangleq \text{cov}\left[\underline{x}(t);\underline{x}(t)\right] \tag{25}$$

$$= E\left\{\left(\underline{x}(t) - \bar{\underline{x}}(t)\right)\left(\underline{x}(t) - \bar{\underline{x}}(t)\right)'\right\}$$

$$= E\left\{\tilde{\underline{x}}(t)\,\tilde{\underline{x}}'(t)\right\}$$

- Try to relate $\underline{\Sigma}(t+1)$ to $\underline{\Sigma}(t)$.

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$$\underline{\Sigma}(t+1) = E\left\{ \underline{\tilde{x}}(t+1)\, \underline{\tilde{x}}'(t+1) \right\} \tag{26}$$

$$= E\left\{ \left[\underline{A}(t)\, \underline{\tilde{x}}(t) + \underline{L}(t)\, \underline{\tilde{\xi}}(t) \right] \left[\underline{A}(t)\, \underline{\tilde{x}}(t) + \underline{L}(t)\, \underline{\tilde{\xi}}(t) \right]' \right\}$$

$$= \underline{A}(t)\, \underbrace{E\left\{ \underline{\tilde{x}}(t)\, \underline{\tilde{x}}'(t) \right\}}_{\underline{\Sigma}(t)}\, \underline{A}'(t) + \underline{L}(t)\, \underbrace{E\left\{ \underline{\tilde{\xi}}(t)\, \underline{\tilde{\xi}}'(t) \right\}}_{\underline{\Xi}(t)}\, \underline{L}'(t)$$

$$+ \underline{A}(t)\, \underbrace{E\left\{ \underline{\tilde{x}}(t)\, \underline{\tilde{\xi}}'(t) \right\}}_{\underline{0}}\, \underline{L}'(t) + \underline{L}(t)\, \underbrace{E\left\{ \underline{\tilde{\xi}}(t)\, \underline{\tilde{x}}'(t) \right\}}_{\underline{0}}\, \underline{A}'(t)$$

State $\underline{x}(t)$ and error $\underline{\tilde{x}}(t)$ independent of $\underline{\xi}(t)$, since $\underline{\xi}(t)$ is discrete-time WN.

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- Hence

$$\boxed{\begin{aligned} \underline{\Sigma}(t+1) &= \underline{A}(t)\, \underline{\Sigma}(t)\, \underline{A}'(t) \\ &\quad + \underline{L}(t)\, \underline{\Xi}(t)\, \underline{L}'(t) \end{aligned}} \tag{27}$$

- Note: Covariance Propagation independent of input $\underline{u}(t)$.

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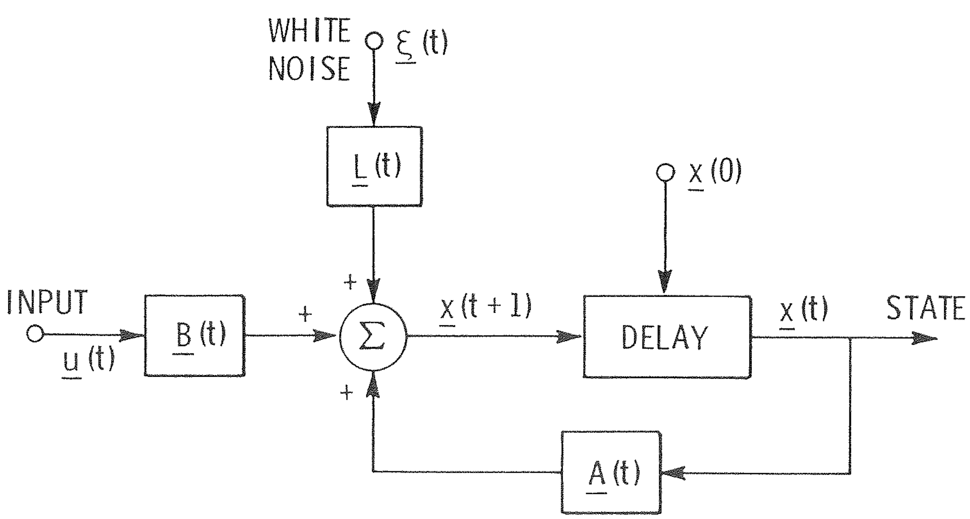
STOCHASTIC DYNAMIC SYSTEM



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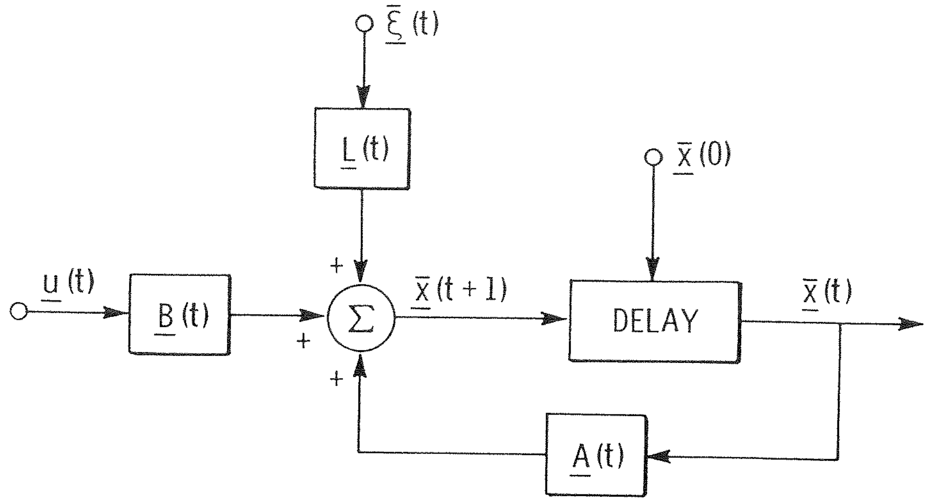
DYNAMIC SYSTEM THAT GENERATES MEAN STATE

Replica of system dynamics



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CONCLUSION

- The mean $\bar{\underline{x}}(t)$ of $\underline{x}(t)$ is generated by a deterministic dynamic system with identical structure to the stochastic one. All random quantities are replaced by their expected values.

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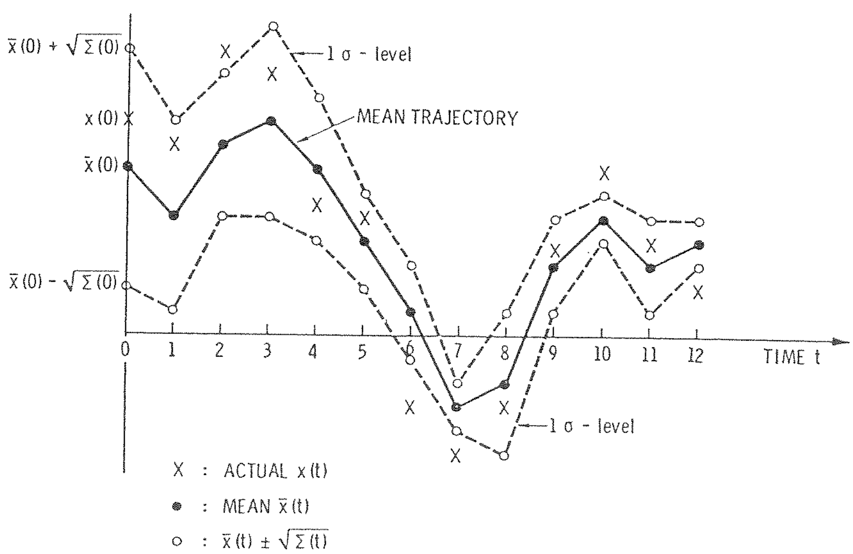

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STEADY-STATE CONSIDERATIONS

- Time Invariant State Dynamics

$$\underline{x}(t+1) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) + \underline{L}\,\underline{\xi}(t) \qquad (28)$$

- Stationary white noise

$$\operatorname{cov}\left[\underline{\xi}(t);\underline{\xi}(\tau)\right] = \underline{\Xi}\,\delta_{t\tau} \qquad (29)$$

- Is there a limiting behavior of $\underline{\Sigma}(t)$?

Steady-state (statistical)
Effects of initial
conditions die-out
for stable plants

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RESULT

- If $\underline{A}$ is strictly stable matrix, i.e.

$$\left|\lambda_i(\underline{A})\right| < 1; \quad i = 1, 2, \ldots, n \qquad (30)$$

- Then for all $\underline{\Sigma}(0)$

$$\lim_{t\to\infty} \underline{\Sigma}(t) = \underline{\Sigma} \quad \text{exists} \qquad (31)$$

$$\underline{\Sigma} = \underline{\Sigma}' \geq \underline{0} = \text{constant} \qquad (32)$$

$$\boxed{\underline{\Sigma} = \underline{A}\,\underline{\Sigma}\,\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}'} \qquad (33)$$

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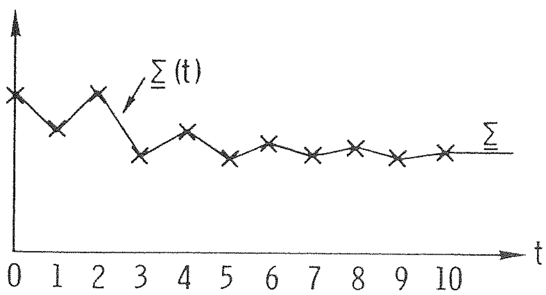
- If $[\underline{A}, \underline{L}]$ is controllable, then $\underline{\Sigma}$ is positive definite.



Recall
$[\underline{A}, \underline{L}]$ = controllable iff
$\text{rank}\left[\underline{L} \,\vdots\, \underline{A}\,\underline{L} \,\vdots\, \underline{A}^2\underline{L} \,\vdots \cdots \vdots\, \underline{A}^{n-1}\underline{L}\right] = n$
i.e. WN excites all
modes of dynamic system

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FURTHER FACTS

- If initial state $\underline{x}(0)$ is gaussian.
- If white noise $\underline{\xi}(t)$ is gaussian for all t.
- Then $\underline{x}(t)$ is gaussian for all $t > 0$.

God made linear systems
and gaussian processes
together.