Approximation of Continuous-Time Linear Stochastic Systems by Discrete-Time Equivalents

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Last Revised: November 3, 2001

Ref. No. KF#4-A

Theme

- We shall only discuss the case of linear time-invariant systems
  - extensions to the time-varying case are available, [1], [2]
- The Basic Problem:
  - we start with a continuous-time linear dynamic system driven by stationary continuous-time white noise
  - we want to correctly simulate it on a digital computer using a discrete-time equivalent
  - how do we determine the “correct” linear discrete-time system and its noise statistics, so that both systems have identical statistical properties at the discrete time instants?

Problem Formulation

CONTINUOUS LTI SYSTEM

(1) $\frac{dx(t)}{dt} = Ax(t) + L\xi(t)$

(2) $E\{x(0)\} \equiv \bar{x}_0;\ cov[x(0; x(0] \equiv \Sigma_0$

$\xi(t)$: continuous white noise

(3) $E\{\xi(t)\} = 0;\ cov[\xi(t); \xi(\tau)] = \Xi\delta(t-\tau)$

- Simulation time step: $\Delta = t_{k+1} - t_k$

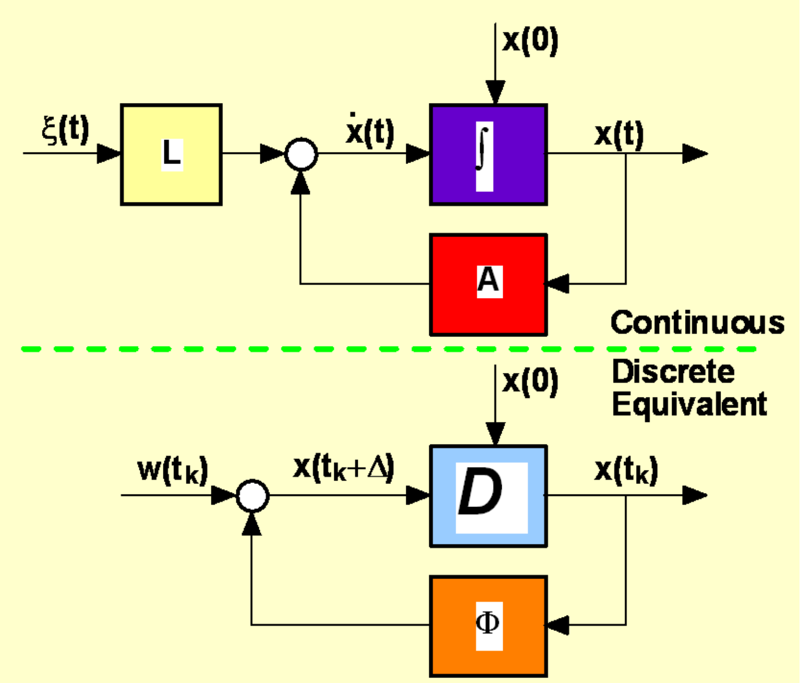
DISCRETE-TIME EQUIVALENT $(t_0 = 0)$

(4) $x(t_k + \Delta) = \Phi x(t_k) + w(t_k)$

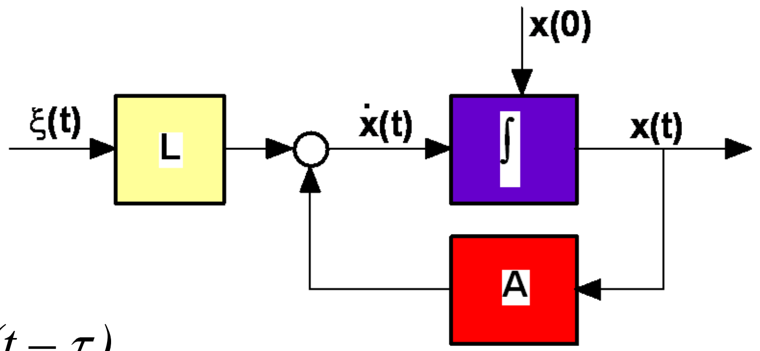
$w(t_k)$: discrete white noise

- Find the state-transition matrix Φ
- Find the mean, $E\{w(t_k)\}$
- Find the covariance matrix,

$cov[w(t_k); w(t_j)] \equiv Q\delta_{t_k t_j}$



Continuous-Time Stochastic Dynamics



(5) $$\frac{dx(t)}{dt} = Ax(t) + L\xi(t)$$

(6) $E\{x(0)\} \equiv \bar{x}_0;\ cov[x(0; x(0] \equiv \Sigma_0$

(7) $E\{\xi(t)\} = 0;\ cov[\xi(t); \xi(\tau)] = \Xi\delta(t-\tau)$

- Evolution of mean state, $\bar{x}(t) \equiv E\{x(t)\}$

(8) $$\frac{d\bar{x}(t)}{dt} = A\bar{x}(t); \quad \bar{x}(0) = \bar{x}_0 \quad \Rightarrow \quad \bar{x}(t) = e^{At}\bar{x}_0$$

- Evolution of covariance matrix, $\Sigma(t) \equiv cov[x(t); x(t)]$

(9) $$\frac{d\Sigma(t)}{dt} = A\Sigma(t) + \Sigma(t)A' + L\Xi L'; \quad \Sigma(0) = \Sigma_0 \quad \Rightarrow$$

(10) $$\Sigma(t) = e^{At}\Sigma_0 e^{A't} + \int_0^t e^{A(t-\tau)} L\Xi L' e^{A'(t-\tau)} d\tau$$

Discrete-Time Stochastic Dynamics

- Simulation time - step:

(11) $\Delta = t_{k+1} - t_k; \quad k = 0, 1, 2, \ldots; \quad t_0 \equiv 0$

- Notation: $x(k) \equiv x(t_k); \quad w(k) \equiv w(t_k)$
- Structure of discrete LTI system

(12) $x(t_{k+1}) = x(t_k + \Delta) = \Phi x(t_k) + w(t_k) \Rightarrow$

(13) $x(k+1) = \Phi x(k) + w(k)$

- Mean propagation: $\bar{x}(k) \equiv E\{x(k)\}$

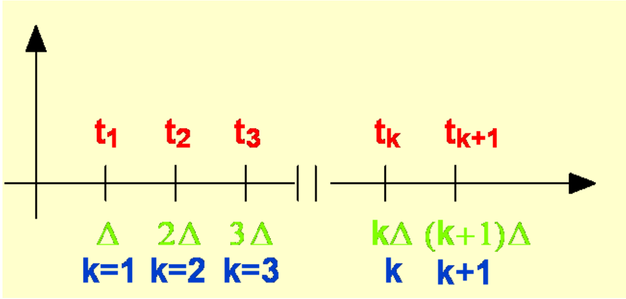
(14) $\bar{x}(k+1) = \Phi \bar{x}(k) + \bar{w}(k); \quad \bar{x}(0) = \bar{x}_0$

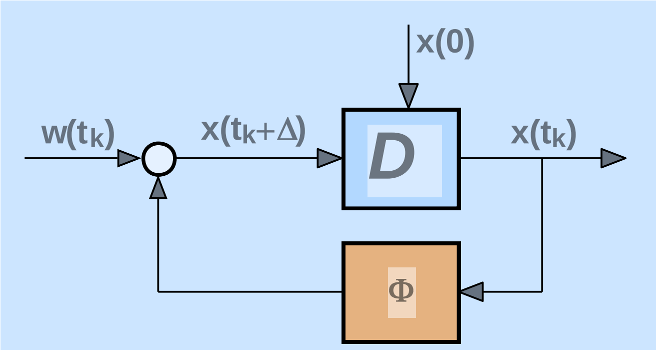
- Covariance : $P(k) \equiv cov[x(k); x(k)]$

(15) $P(k+1) = \Phi P(k) \Phi' + Q, \quad P(0) = \Sigma_0$

where

(16) $cov[w(k); w(j)] \equiv Q \delta_{kj}$





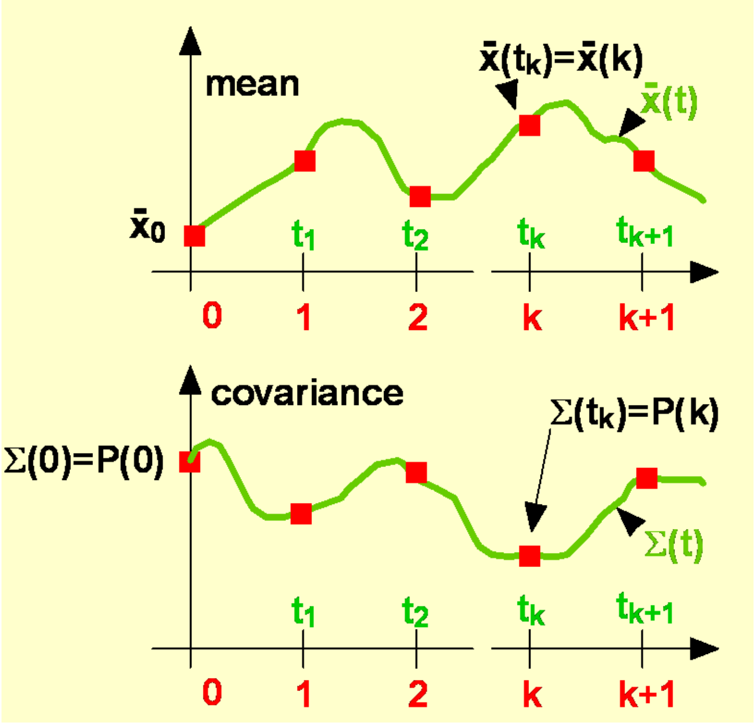
Basic Problem

- Determine $\Phi, Q, \overline{w}(k)$ in the discrete-time model such that at the discrete time instants the mean and covariance of the state of continuous and discrete models are identical
- We shall show that

(17) $\Phi = e^{A\Delta}$

(18) $\overline{w}(k) = 0$

(19) $Q = \int_0^{\Delta} e^{A(\Delta-\tau)} L \Xi L' e^{A'(\Delta-\tau)} d\tau$



Matching The Mean-State

- The continuous-time mean-state, $\bar{x}(t) \equiv E\{x(t)\}$, is given by eq. (8)

(20) $$\bar{x}(t) = e^{At}\bar{x}_0$$

- Set $t = \Delta,$ to obtain

(21) $$\bar{x}(\Delta) = e^{A\Delta}\bar{x}_0$$

- At $t = \Delta$ the discrete-time index is $k = 1,$ so from the discrete-time mean-state eq. (14) we have

(22) $$\bar{x}(1) = \Phi\bar{x}(0) + \overline{w}(0) = \Phi\bar{x}_0 + \overline{w}(0) \quad \Rightarrow \quad \Phi = e^{A\Delta}, \quad \overline{w}(0) = 0$$

- By induction, at times t_k ,discrete-index $k,$ we conclude

(23) $$\Phi = e^{A\Delta}, \quad \overline{w}(k) = 0 \quad \Rightarrow \quad \bar{x}(k\Delta) = \bar{x}(k) \quad \forall k = 0,1,2,\ldots$$

so that the discrete-time system evolves as

(24) $$x(k+1) = e^{A\Delta}x(k) + w(k); \quad \bar{x}(0) = \bar{x}_0; \quad \overline{w}(k) = 0;$$

$$cov[w(k); w(j)] = Q\delta_{kj}$$

Matching Covariance Matrices

- Let $t = \Delta$ so that $k = 1$. From the continuous-time covariance propagation, eq. (10), the state covariance is

(25) $$\Sigma(\Delta) = e^{A\Delta}\Sigma_0 e^{A'\Delta} + \int_0^{\Delta} e^{A(\Delta-\tau)} L \Xi L' e^{A'(\Delta-\tau)} d\tau$$

- From eq. (15), with $\Phi = e^{A\Delta}$, the discrete-system covariance is

(26) $$P(1) = \Phi P(0)\Phi' + Q = e^{A\Delta}\Sigma_0 e^{A'\Delta} + Q$$

- From eqs. (25) and (26), requiring $\Sigma(\Delta) = P(1)$, we conclude

(27) $$Q = \int_0^{\Delta} e^{A(\Delta-\tau)} L \Xi L' e^{A'(\Delta-\tau)} d\tau$$

  and by induction, eq. (27) guarantees that

(28) $$\Sigma(k\Delta) = P(k) \quad \forall k = 0, 1, 2, \ldots$$

- Note that even if $L \Xi L'$ is a diagonal matrix, the equivalent discrete-time covariance, Q, is not diagonal, in general

Steady-State Calculations

- Suppose that the continuous - time system is stable, i.e.

(29) $$Re\{\lambda_i(A)\} < 0 \quad \Rightarrow \quad |\lambda_i(\Phi)| = |\lambda_i(e^{A\Delta})| < 1 \quad \forall i = 1, 2, \ldots, n$$

- For stable systems we know that the steady - state covariance matrices, Σ and P, satisfy the Lyapunov equations

(30) $$0 = A\Sigma + \Sigma A' + L\Xi L' \; ; \quad P = e^{A\Delta} P e^{A'\Delta} + Q$$

- It follows that we can calculate the equivalent covariance P without solving the convolution matrix integral (27) as follows

  STEP#1: Calculate, using MATLAB, Σ from eq. (30)

  STEP#2: Since we must have $\Sigma = P,$ from eq. (30) we obtain

(31) $$\Sigma = e^{A\Delta} \Sigma e^{A'\Delta} + Q \quad \Rightarrow \quad Q = \Sigma - e^{A\Delta} \Sigma e^{A'\Delta}$$

- Recall that the matrix exponential is given by the infinite series

(32) $$e^{A\Delta} = I + A\Delta + \frac{1}{2!} A^2 \Delta^2 + \ldots + \frac{1}{k!} A^k \Delta^k + \ldots$$

- For small $\Delta,\ \Delta \to 0,$ ignoring the quadratic and higher - order terms $e^{A\Delta} \cong I + A\Delta,$ and so we can use the approximation

(33) $$Q \cong \Delta(L\Xi L')$$

Concluding Remarks

- We have provided the precise manner by which we can simulate a continuous-time linear time-invariant (LTI) system using a discrete-time equivalent, in the sense that
  - the expected value of the state vector is identical at the sampling times for both systems
  - the state covariance matrix is identical at the sampling times for both systems
- Naturally, it makes no sense of having the real state vectors be identical, only their statistical properties

References

[1]. A.Gelb, *Applied Optimal Estimation*, MIT Press, 1974, pp. 72-75

[2]. M.S. Grewal and A.P. Andrews, *Kalman Filtering: Theory and Practice*, Prentice-Hall, 1993, pp. 88-91