REF. NO. KF # 4

1
STOCHASTIC ESTIMATION

Response of Linear Systems to White Noise Inputs: Continuous Time Case

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PROBLEM FORMULATION

- State Dynamics:

deterministic input

continuous-time WN

$$\boxed{\begin{aligned}\dot{\underline{x}}(t) &= \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t)\\ &\quad + \underline{L}(t)\,\underline{\xi}(t)\end{aligned}} \qquad (1)$$

- Initial State $\underline{x}(t_0)$: random vector
- $\underline{\xi}(t)$: continuous white noise
- $\underline{A}(t)$, $\underline{B}(t)$, $\underline{L}(t)$, $\underline{u}(t)$: deterministic

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BASIC QUESTIONS

- How does $\underline{x}(t)$ propagate on the average?

  ⟶ Determine $E\left\{\underline{x}(t)\right\} \stackrel{\Delta}{=} \bar{\underline{x}}(t)$

- How variable is $\underline{x}(t)$ about its average?

  ⟶ Determine $\mathrm{cov}\left[\underline{x}(t);\underline{x}(t)\right] \stackrel{\Delta}{=} \underline{\Sigma}(t)$

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INITIAL UNCERTAINTY

- $\underline{x}(t_0)$ is random vector

$$\underline{x}(t_0) \in R_n \qquad (2)$$

$$E\{\underline{x}(t_0)\} = \underline{\bar{x}}(t_0) \text{ assumed known} \qquad (3)$$

$\underline{\Sigma}_0$ = initial state uncertainty

$$\operatorname{cov}[\underline{x}(t_0); \underline{x}(t_0)] \qquad (4)$$

$$= E\left\{\left(\underline{x}(t_0) - \underline{\bar{x}}(t_0)\right)\left(\underline{x}(t_0) - \underline{\bar{x}}(t_0)\right)'\right\}$$

$$= \underline{\Sigma}_0 \qquad \text{assumed known}$$

$$\underline{\Sigma}_0 = \underline{\Sigma}_0' \geq \underline{0} \qquad (5)$$

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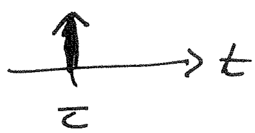
DRIVING WHITE NOISE

$\underline{\xi}(t)$ white continuous-time noise

$$E\{\underline{\xi}(t)\} \overset{\Delta}{=} \underline{\bar{\xi}}(t) \qquad (6)$$

$$\operatorname{cov}[\underline{\xi}(t); \underline{\xi}(\tau)] = \underline{\Xi}(t)\,\delta(t-\tau) \qquad (7)$$

Dirac "delta" function



$$\underline{\Xi}(t) = \underline{\Xi}'(t) \geq \underline{0} \qquad (8)$$

infinite variance
no time-correlation

$\underline{\xi}(t)$, $\underline{x}(t_0)$ independent for all t

$$\rightarrow \quad \operatorname{cov}[\underline{\xi}(t); \underline{x}(t_0)] = \underline{0} \qquad (9)$$

$\underline{\bar{\xi}}(t)$, $\underline{\Xi}(t)$ assumed known.

called intensity matrix

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SUMMARY OF RESULTS

- State $\underline{x}(t) \in R_n$ is a colored Markov process (not white).
- Mean Dynamics

$$E\{\underline{x}(t)\} \overset{\Delta}{=} \underline{\bar{x}}(t) \qquad (10)$$

$$\boxed{\frac{d}{dt}\underline{\bar{x}}(t) = \underline{A}(t)\underline{\bar{x}}(t) + \underline{B}(t)\underline{u}(t) + \underline{L}(t)\underline{\bar{\xi}}(t)} \qquad (11)$$

$$\underline{\bar{x}}(t_0) = E\{\underline{x}(t_0)\} \text{ known} \qquad (12)$$

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- $\bar{x}(t)$ is computed either by direct numerical solution of (11) or by the formula

$$\underline{\bar{x}}(t) = \underline{\Phi}(t, t_0)\,\underline{\bar{x}}(t_0) + \int_{t_0}^{t} \underline{\Phi}(t, \tau)\,\underline{B}(\tau)\,\underline{u}(\tau)\,d\tau + \int_{t_0}^{t} \underline{\Phi}(t, \tau)\,\underline{L}(\tau)\,\underline{\bar{\xi}}(\tau)\,d\tau \qquad (13)$$

- This is standard convolution for LTV systems.
- $\underline{\Phi}(t, \tau)$ is fundamental transition matrix

$$\frac{d}{dt}\underline{\Phi}(t,\tau) = \underline{A}(t)\,\underline{\Phi}(t,\tau)$$

$$\underline{\Phi}(\tau, \tau) = \underline{I}$$

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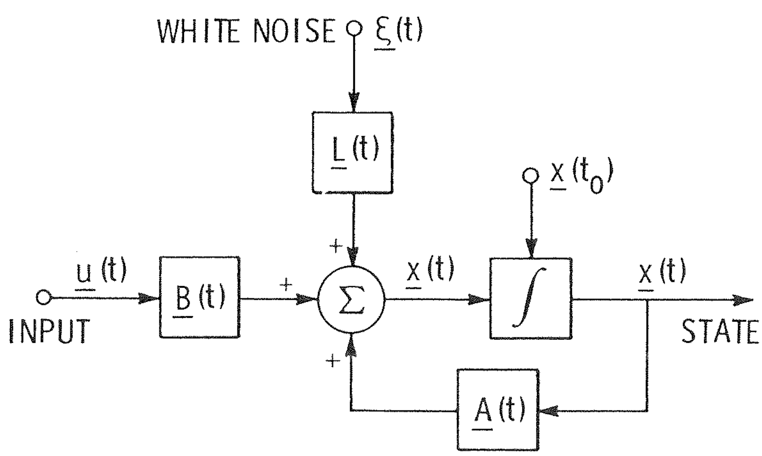
STOCHASTIC DYNAMIC SYSTEM



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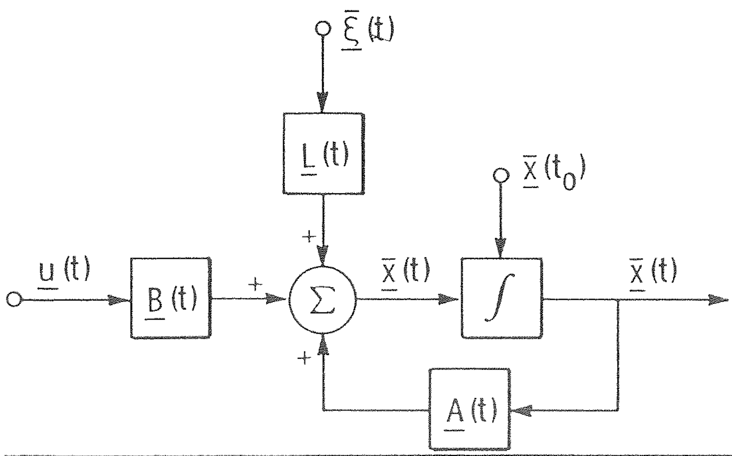
DYNAMIC SYSTEM THAT GENERATES MEAN STATE

Replica of system dynamics



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- Covariance Dynamics

$$\underline{\Sigma}(t) \triangleq E\left\{\left(\underline{x}(t) - \underline{\bar{x}}(t)\right)\left(\underline{x}(t) - \underline{\bar{x}}(t)\right)'\right\} \tag{14}$$
$$= \text{cov}\left[\underline{x}(t);\ \underline{x}(t)\right]$$

independent of $\underline{B}(t)$, $\underline{u}(t)$!

$$\underline{\Sigma}(t) = \underline{\Sigma}'(t) \geq \underline{0} \tag{15}$$

$$\boxed{\frac{d}{dt}\underline{\Sigma}(t) = \underline{A}(t)\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{A}'(t) + \underline{L}(t)\underline{\Xi}(t)\underline{L}'(t)} \tag{16}$$

Dynamic evolution of covariance matrix

LTV matrix differential eq. of Lyapunov type

$$\underline{\Sigma}(t_0) = \underline{\Sigma}_0 = \text{cov}\left[\underline{x}(t_0);\ \underline{x}(t_0)\right] \tag{17}$$

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$\underline{\Sigma}(t)$ is determined either from direct numerical solution of Eq (16) or by the formula

$$\underline{\Sigma}(t) = \underline{\Phi}(t, t_0)\underline{\Sigma}(t_0)\underline{\Phi}'(t, t_0) + \int_{t_0}^{t} \underline{\Phi}(t, \tau)\underline{L}(\tau)\underline{\Xi}(\tau)\underline{L}'(\tau)\underline{\Phi}'(t, \tau)\, d\tau \tag{18}$$

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- Cross Covariance Dynamics (Correlation Matrix)

$$\underline{\Sigma}(t, \tau) \triangleq E\left\{\left(\underline{x}(t) - \underline{\bar{x}}(t)\right)\left(\underline{x}(\tau) - \underline{\bar{x}}(\tau)\right)'\right\}$$
$$= \text{cov}\left[\underline{x}(t);\ \underline{x}(\tau)\right]; \quad \tau < t \tag{19}$$

- Define

$$\underline{\Sigma}(t, t) \triangleq \underline{\Sigma}(t) \tag{20}$$

$$\frac{d}{dt}\underline{\Phi}(t, \tau) = \underline{A}(t)\underline{\Phi}(t, \tau)$$
$$\underline{\Phi}(\tau, \tau) = \underline{I}$$

- Given $\underline{\Sigma}(\tau, \tau)$, $\underline{\Sigma}(t, \tau)$ is computed by

$$\boxed{\underline{\Sigma}(t, \tau) = \underline{\Sigma}(\tau, \tau)\underline{\Phi}'(t, \tau); \quad t \geq \tau} \tag{21}$$

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DIGRESSION

For proofs we need certain properties of expectation operator

$$E\left\{\underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{y}(t)\right\} \qquad (22)$$
$$= \underline{A}(t)\,E\left\{\underline{x}(t)\right\} + \underline{B}(t)\,E\left\{\underline{y}(t)\right\}$$

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$$\frac{d}{dt}E\left\{\underline{x}(t)\right\} = E\left\{\frac{d}{dt}\underline{x}(t)\right\} \qquad (23)$$

$$\int_a^b E\left\{\underline{x}(t)\right\}dt = E\left\{\int_a^b \underline{x}(t)\,dt\right\} \qquad (24)$$

Expection operator commutes with differentiation and integration operators

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DERIVATION OF MEAN DYNAMICS

- State Dynamics

$$\dot{\underline{x}}(t) = \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\underline{\xi}(t) \qquad (25)$$

$$E\left\{\dot{\underline{x}}(t)\right\} = \frac{d}{dt}E\left\{\underline{x}(t)\right\} \qquad (26)$$
$$= E\left\{\underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\underline{\xi}(t)\right\}$$
$$= \underline{A}(t)\,E\left\{\underline{x}(t)\right\} + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,E\left\{\underline{\xi}(t)\right\}$$

- Therefore

$$\boxed{\dot{\bar{\underline{x}}}(t) = \underline{A}(t)\,\bar{\underline{x}}(t) + \underline{B}(t)\,\underline{u}(t) + \underline{L}(t)\,\bar{\underline{\xi}}(t)} \qquad (27)$$

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DEFINITION OF "ERROR" VECTOR $\tilde{\underline{x}}(t)$ (prediction error)

$$\left.\begin{array}{l} \tilde{\underline{x}}(t) \triangleq \underline{x}(t) - \bar{\underline{x}}(t) \\ E\left\{\tilde{\underline{x}}(t)\right\} = \underline{0} \end{array}\right\} \qquad (28)$$

NOTE: From Eqs (25) and (27)

$$\boxed{\dot{\tilde{\underline{x}}}(t) = \underline{A}(t)\,\tilde{\underline{x}}(t) + \underline{L}(t)\,\tilde{\underline{\xi}}(t)} \qquad (29)$$

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$$\tilde{\underline{\xi}}(t) \triangleq \underline{\xi}(t) - \bar{\underline{\xi}}(t) \tag{30}$$

$$E\left\{\tilde{\underline{\xi}}(t)\right\} = \underline{0} \tag{31}$$

$$\text{cov}\left[\tilde{\underline{\xi}}(t);\ \tilde{\underline{\xi}}(\tau)\right] = E\left\{\tilde{\underline{\xi}}(t)\ \tilde{\underline{\xi}}'(\tau)\right\} \tag{32}$$

$$= \text{cov}\left[\underline{\xi}(t);\ \underline{\xi}(\tau)\right] = \underline{\Xi}(t)\ \delta(t-\tau)$$

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- Hence

$$\underline{\Sigma}(t) = \text{cov}\left[\underline{x}(t);\ \underline{x}(t)\right] \tag{33}$$

$$= E\left\{\left(\underline{x}(t) - \bar{\underline{x}}(t)\right)\left(\underline{x}(t) - \bar{\underline{x}}(t)\right)'\right\}$$

$$= E\left\{\tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t)\right\}$$

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DERIVATION OF COVARIANCE DYNAMICS

$$\frac{d}{dt}\underline{\Sigma}(t) = \frac{d}{dt} E\left\{\tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t)\right\} \tag{34}$$

$$= E\left\{\frac{d}{dt}\left(\tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t)\right)\right\}$$

$$= E\left\{\dot{\tilde{\underline{x}}}(t)\ \tilde{\underline{x}}'(t) + \tilde{\underline{x}}(t)\ \dot{\tilde{\underline{x}}}'(t)\right\}$$

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- From Eqs (29) and (34)

$$\dot{\underline{\Sigma}}(t) = E\left\{\underline{A}(t)\ \tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t) + \underline{L}(t)\ \tilde{\underline{\xi}}(t)\ \tilde{\underline{x}}'(t)\right. \tag{35}$$

$$\left. + \tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t)\ \underline{A}'(t) + \tilde{\underline{x}}(t)\ \tilde{\underline{\xi}}'(t)\ \underline{L}'(t)\right\}$$

$$= \underline{A}(t)\ \underbrace{E\left\{\tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t)\right\}}_{\underline{\Sigma}(t)} + \underbrace{E\left\{\tilde{\underline{x}}(t)\ \tilde{\underline{x}}'(t)\right\}}_{\underline{\Sigma}(t)}\ \underline{A}'(t)$$

$$+ \underline{L}(t)\ E\left\{\tilde{\underline{\xi}}(t)\ \tilde{\underline{x}}'(t)\right\} + E\left\{\tilde{\underline{x}}(t)\ \tilde{\underline{\xi}}'(t)\right\}\ \underline{L}'(t)$$

- Need to calculate

$$E\left\{\tilde{\underline{\xi}}(t)\ \tilde{\underline{x}}'(t)\right\} = \left[E\left\{\tilde{\underline{x}}(t)\ \tilde{\underline{\xi}}'(t)\right\}\right]' \tag{36}$$

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$$E\left\{\tilde{\underline{\xi}}(t)\,\tilde{\underline{x}}'(t)\right\} \qquad (37)$$

$$= E\left\{\tilde{\underline{\xi}}(t)\left[\underline{\Phi}(t,t_0)\,\tilde{\underline{x}}(t_0) + \int_{t_0}^{t}\underline{\Phi}(t,\tau)\,\underline{L}(\tau)\,\tilde{\underline{\xi}}(\tau)\,d\tau\right]'\right\}$$

$$= \underbrace{E\left\{\tilde{\underline{\xi}}(t)\,\tilde{\underline{x}}'(t_0)\right\}}_{\underline{0}}\,\underline{\Phi}'(t,t_0)$$

because initial state is independent of WN

$$+ E\left\{\tilde{\underline{\xi}}(t)\int_{t_0}^{t}\tilde{\underline{\xi}}(\tau)\,\underline{L}'(\tau)\,\underline{\Phi}'(t,\tau)\,d\tau\right\}$$

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$$= \int_{t_0}^{t} E\left\{\tilde{\underline{\xi}}(t)\,\tilde{\underline{\xi}}'(\tau)\right\}\underline{L}'(\tau)\,\underline{\Phi}'(t,\tau)\,d\tau$$

$$= \int_{t_0}^{t}\underline{\Xi}(t)\,\delta(t-\tau)\,\underline{L}'(\tau)\,\underline{\Phi}'(t,\tau)\,d\tau$$

$$= \frac{1}{2}\,\underline{\Xi}(t)\,\underline{L}'(t)\underbrace{\underline{\Phi}'(t,t)}_{\underline{I}} = \frac{1}{2}\,\underline{\Xi}(t)\,\underline{L}'(t)$$

Digression

$$\int_a^b f(\tau)\,\delta(\tau-x)\,d\tau = f(x)$$

$$a < x < b$$

$$\int_a^b f(\tau)\,\delta(\tau-b)\,d\tau = \frac{1}{2}f(b)$$

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- From (35) and (37)

$$\dot{\underline{\Sigma}}(t) = \underline{A}(t)\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\,\underline{A}'(t) + \frac{1}{2}\,\underline{L}(t)\,\underline{\Xi}(t)\,\underline{L}'(t) + \left[\frac{1}{2}\,\underline{\Xi}(t)\,\underline{L}'(t)\right]'\underline{L}'(t) \qquad (38)$$

or

LTV Lyapunov matrix DE

$$\boxed{\dot{\underline{\Sigma}}(t) = \underline{A}(t)\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\,\underline{A}'(t) + \underline{L}(t)\,\underline{\Xi}(t)\,\underline{L}'(t)} \qquad (39)$$

independent of $\underline{B}(t)$, $\underline{u}(t)$

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STEADY STATE CONSIDERATIONS

- Time Invariant State Dynamics

$$\underline{x}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) + \underline{L}\,\underline{\xi}(t) \qquad (40)$$

- Stationary White Noise

$$\mathrm{cov}\left[\underline{\xi}(t);\underline{\xi}(\tau)\right] = \underline{\Xi}\,\delta(t-\tau) \qquad (41)$$

Constant intensity matrix

- Is there a limiting behavior of $\underline{\Sigma}(t)$?

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RESULT

- If $\underline{A}$ is strictly stable matrix, i.e.

$$\mathrm{Re}\quad \lambda_i(\underline{A}) < 0; \quad i = 1, 2, \ldots, n \qquad (42)$$

- Then for all $\underline{\Sigma}(t_0)$

$$\lim_{t\to\infty} \underline{\Sigma}(t) = \underline{\Sigma} \quad \text{exists} \qquad (43)$$

$$\underline{\Sigma} = \underline{\Sigma}' \geq \underline{0} = \text{constant} \qquad (44)$$

$$\boxed{\underline{0} = \underline{A}\,\underline{\Sigma} + \underline{\Sigma}\,\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}'} \qquad (45)$$

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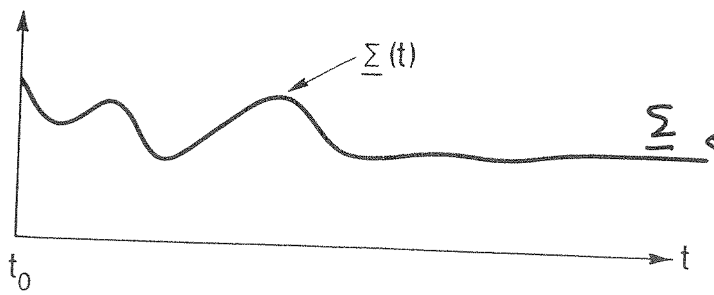
- If $[\underline{A}, \underline{L}]$ is controllable, then $\underline{\Sigma}$ is positive definite.

Still solution of

$$\dot{\underline{\Sigma}}(t) = \underline{A}\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}'$$

$$\underline{\Sigma}'(t_0) = \underline{\Sigma}_0$$



steady-state covariance matrix

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FURTHER FACTS

- If initial state $\underline{x}(t_0)$ is gaussian.
- If white noise $\underline{\xi}(t)$ is gaussian.
- Then, $\underline{x}(t)$ is gaussian for all $t > 0$.

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NUMERICAL EXAMPLE

- State Dynamics

$$\dot{x}(t) = -x(t) + \xi(t) \tag{46}$$

$$\left.\begin{aligned} &E\{x(0)\} = 2 \\ &\operatorname{cov}[x(0);\, x(0)] = 4 \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} &E\{\xi(t)\} = 0 \\ &\operatorname{cov}[\xi(t);\, \xi(\tau)] = 6\,\delta(t-\tau) \end{aligned}\right\} \tag{48}$$

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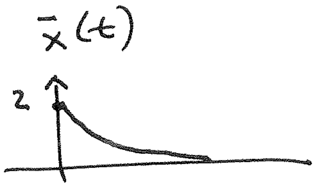
- Evolution of Mean

$$\bar{x}(t) = E\{x(t)\} \tag{49}$$

$$\dot{\bar{x}}(t) = -\bar{x}(t)\,; \qquad \bar{x}(0) = 2 \tag{50}$$

$$\boxed{\bar{x}(t) = 2\,e^{-t}} \tag{51}$$



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- Evolution of Variance

$$\Sigma(t) = E\left\{\left(x(t) - \bar{x}(t)\right)^2\right\} \tag{52}$$

$$\dot{\Sigma}(t) = -2\,\Sigma(t) + 6\,; \quad \Sigma(0) = 4 \tag{53}$$

$$\boxed{\Sigma(t) = e^{-2t} + 3} \tag{54}$$