REF NO. KF #6

1
STOCHASTIC ESTIMATION

The Discrete-Time Kalman Filter

Part 1

2
MOTIVATION

- Given a discrete-time, linear, time-varying plant
  - with random initial state
  - driven by white plant noise
- Given noisy measurements of linear combinations of the plant state variables
- Determine "best" estimate of the plant state variables

3
PROBLEM FORMULATION

- State Dynamics

$$\boxed{\begin{aligned}\underline{x}(t+1) &= \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t)\\ &+\underline{L}(t)\underline{\xi}(t)\end{aligned}} \qquad (1)$$

- Measurement Equation

$$\boxed{\begin{aligned}\underline{z}(t+1) &= \underline{C}(t+1)\underline{x}(t+1)\\ &+\underline{\theta}(t+1)\end{aligned}} \qquad (2)$$

- Time index: $t=0, 1, 2, \ldots$

4

VARIABLE DEFINITIONS

$\underline{x}(t) \in R_n$	state vector (stochastic sequence non-white)
$\underline{u}(t) \in R_m$	deterministic input sequence
$\underline{\xi}(t) \in R_p$	white plant noise
$\underline{\theta}(t) \in R_r$	white measurement noise
$\underline{z}(t) \in R_r$	measurement vector

5

PROBABILISTIC INFORMATION

- Initial State $\underline{x}(o)$ is gaussian

$$E\left\{\underline{x}(o)\right\} = \bar{\underline{x}}(o) \qquad (3)$$

$$\mathrm{cov}\left[\underline{x}(o);\underline{x}(o)\right] = \underline{\Sigma}_0 = \underline{\Sigma}_0' \geq \underline{0} \qquad (4)$$

6

- plant noise $\underline{\xi}(t)$ is gaussian discrete white noise

$$E\left\{\underline{\xi}(t)\right\} = \underline{0} \qquad (5)$$

$$\mathrm{cov}\left[\underline{\xi}(t);\ \underline{\xi}(\tau)\right] = \underline{\Xi}(t)\,\delta_{t\tau} \qquad (6)$$

$$\underline{\Xi}(t) = \underline{\Xi}'(t) \geq \underline{0} \qquad (7)$$

7

- Measurement noise $\underline{\theta}(t)$ is gaussian discrete white noise

$$E\left\{\underline{\theta}(t)\right\} = \underline{0} \qquad (8)$$

$$\mathrm{cov}\left[\underline{\theta}(t);\ \underline{\theta}(\tau)\right] = \underline{\Theta}(t)\,\delta_{t\tau} \qquad (9)$$

$$\underline{\Theta}(t) = \underline{\Theta}'(t) > \underline{0} \qquad \cancel{10} \qquad (10)$$

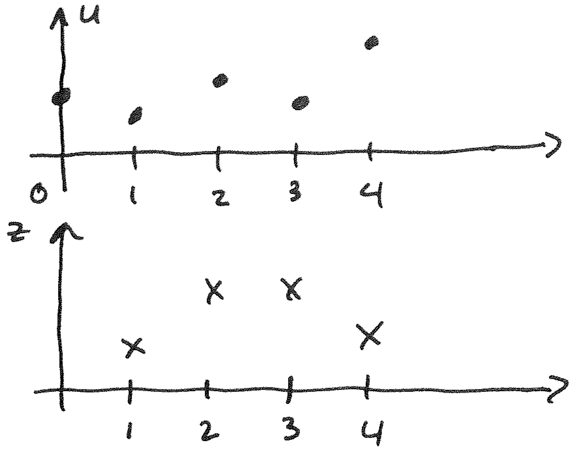
→ every measurement is corrupted by white noise

- $\underline{x}(o)$, $\underline{\xi}(t)$, $\underline{\theta}(\tau)$ (11)

  are independent for all t,τ

8

DEFINITION OF FILTERING PROBLEM



- Let t denote present value of time
- Given the sequence of past inputs

$$U(t-1) \triangleq \{\underline{u}(0), \underline{u}(1), \ldots, \underline{u}(t-1)\} \quad (12)$$

- Given the sequence of past measurements

$$Z(t) \triangleq \{\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(t)\} \quad (13)$$

- Determine a "good" estimate of the state $\underline{x}(t)$

9 FACT!

THE PROPERTY OF THE CONDITIONAL DENSITY FUNCTION

$p(\underline{x}(t)/Z(t), U(t-1))$

The linearity of
- (a) the state equation
- (b) the measurement equation

and the gaussian nature of
- (a) the initial state, $\underline{x}(0)$
- (b) the plant white noise, $\underline{\xi}(t)$
- (c) the measurement white noise $\underline{\theta}(t)$

imply that

$$p(\underline{x}(t)/Z(t), U(t-1)) \text{ is gaussian} \quad (14)$$

10

Hence the conditional state density function is uniquely characterized by

$$\underline{\hat{x}}(t|t) = \int \underline{x}(t)\, p(\underline{x}(t)|Z(t), U(t-1))\, d\underline{x}(t)$$

(i) the conditional mean

$$\underline{\hat{x}}(t/t) \triangleq E\{\underline{x}(t)/Z(t), U(t-1)\} \quad (15)$$

(ii) the conditional covariance

$$\underline{\Sigma}(t/t) = \text{cov}[\underline{x}(t); \underline{x}(t)/Z(t), U(t-1)] \quad (16)$$

$$\underline{\Sigma}(t|t) = \int (\underline{x}(t) - \underline{\hat{x}}(t|t))(\underline{x}(t) - \underline{\hat{x}}(t|t))'\, p(\underline{x}(t)|Z(t), U(t-1))\, d\underline{x}(t)$$

11
DEFINITION OF OPTIMAL ESTIMATE OF $\underline{x}(t)$

Since $p(\underline{x}(t)/Z(t), U(t-1)$ is gaussian then all reasonable estimates (mean, median, most probable) are the same.

12
Hence the optimal estimate of $\underline{x}(t)$ given

- past measurements, $Z(t)$
- past inputs, $U(t-1)$

is taken to be the conditional mean

$$\boxed{\hat{\underline{x}}(t/t) = E\left\{\underline{x}(t)/Z(t),\ U(t-1)\right\}} \qquad (17)$$

13
TIME STRUCTURE OF PROBLEM

Recall:

$\underline{x}(t+1) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{L}(t)\underline{\xi}(t)$

- The development has an inductive flavor. The basic process is as follows:

1) Assume that all relevant quantities are available at time t, $Z(t)$, $U(t-1)$

- Then:

(a) "Nature" applies $\underline{\xi}(t)$

(b) We apply $\underline{u}(t)$

(c) The system moves to state $\underline{x}(t+1)$

(d) We make a measurement $\underline{z}(t+1)$ $\underline{z}(t+1) = \underline{C}(t+1)\underline{x}(t+1) + \underline{\theta}(t+1)$

14
2) We want now to make an estimate of $\underline{x}(t+1)$ based on the expanded information

$Z(t+1) = \left\{Z(t),\ \underline{z}(t+1)\right\}$ Add "newest" measurement $\underline{z}(t+1)$ to $Z(t)$

$U(t) = \left\{U(t-1),\ \underline{u}(t)\right\}$ Add "newest" control $\underline{u}(t)$ to $U(t-1)$

15

- The estimation process is then divided into two parts

(I) PREDICT CYCLE
What can say about $\underline{x}(t+1)$ before we make the measurement $\underline{z}(t+1)$

(II) UPDATE CYCLE

How can we improve our information about $\underline{x}(t+1)$ after we make the measurement $\underline{z}(t+1)$.

16

- STRUCTURE OF INFORMATION
- For Predict Cycle:

We have

$$U(t) = \{\underbrace{\underline{u}(0), \underline{u}(1), \ldots, \underline{u}(t-1)}_{U(t-1)}, \underline{u}(t)\} \tag{18}$$

$$Z(t) = \{\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(t)\} \tag{19}$$

17

- For Update Cycle:

We have

$U(t)$ - same as above (20)

$$Z(t+1) = \{\underbrace{\underline{z}(1), \ldots, \underline{z}(t)}_{Z(t)}, \underline{z}(t+1)\} \tag{21}$$

18

WHAT DO WE NEED TO PROCESS THE MEASUREMENT $\underline{z}(t+1)$?

The key quantity that needs to be evaluated is

$$p\left(\underline{x}(t+1)/Z(t+1),\ U(t)\right) \qquad (22)$$

As far as the measurement $\underline{z}(t+1)$ is concerned, let us view as "prior" information the probability density of $\underline{x}(t+1)$ given $Z(t)$, $U(t)$

$$p\left(\underline{x}(t+1)/Z(t),\ U(t)\right) \qquad (23)$$

19

- Then Bayes rule requires

$$p\left(\underline{x}(t+1)/Z(t+1),\ U(t)\right) = \qquad (24)$$

$$= p\left(\underline{z}(t+1)/\underline{x}(t+1), Z(t),\ U(t)\right) \cdot p\left(\underline{x}(t+1)/Z(t),\ U(t)\right) \Big/ p\left(\underline{z}(t+1)/Z(t),\ U(t)\right)$$

20

- Since the measurement equation

$$\underline{z}(t+1) = \underline{C}(t+1)\ \underline{x}(t+1) + \underline{\theta}(t+1) \qquad (25)$$

is linear, we need to establish the gaussian nature of

$$p\left(\underline{x}(t+1)/Z(t),\ U(t)\right) \qquad (26)$$

$$p\left(\underline{z}(t+1)/\underline{x}(t+1),\ Z(t),\ U(t)\right) \qquad (27)$$

$$p\left(\underline{z}(t+1)/Z(t),\ U(t)\right) \qquad (28)$$

- Also note that $\underline{x}(t+1)$ and $\underline{\theta}(t+1)$ are independent.

21

THE PREDICT CYCLE:
EVALUATION OF $p(\underline{x}(t+1)/Z(t), U(t)$

- Induction hypothesis

$$\boxed{p\left(\underline{x}(t)/Z(t), U(t-1)\right) \text{ is gaussian}} \tag{29}$$

$$\hat{\underline{x}}(t/t) = E\left\{\underline{x}(t)/Z(t), U(t-1)\right\} \text{known} \tag{30}$$

$$\underline{\Sigma}(t/t) = \operatorname{cov}\left[\underline{x}(t); \underline{x}(t)/Z(t), U(t-1)\right] \tag{31}$$

known

22

- System Dynamics

$$\underline{x}(t+1) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{L}(t)\,\underline{\xi}(t) \tag{32}$$

- $\underline{x}(t)/Z(t), U(t-1)$ and $\underline{\xi}(t)$ gaussian and independent

$$\Rightarrow \boxed{p\left(\underline{x}(t+1)/Z(t), U(t)\right) \text{ is gaussian}} \tag{33}$$

23

NOTATION

- One step predicted estimate $\hat{\underline{x}}(t+1/t)$

$$\boxed{\hat{\underline{x}}(t+1/t) \triangleq E\left\{\underline{x}(t+1)/Z(t), U(t)\right\}} \tag{34}$$

- One step predicted covariance:

$\underline{\Sigma}(t+1)/t)$

$$\boxed{\underline{\Sigma}(t+1/t) \triangleq \operatorname{cov}\left[\underline{x}(t+1); \underline{x}(t+1)/Z(t), U(t)\right]} \tag{35}$$

24

CALCULATIONS : One-Step prediction

- Mean

$$\boxed{\hat{\underline{x}}(t+1/t) = \underline{A}(t)\hat{\underline{x}}(t/t) + \underline{B}(t)\underline{u}(t)} \tag{36}$$

- Covariance

$$\boxed{\begin{aligned}\underline{\Sigma}(t+1/t) = {} & \underline{A}(t)\,\underline{\Sigma}(t/t)\,\underline{A}'(t) \\ & + \underline{L}(t)\,\underline{\Xi}(t)\,\underline{L}'(t)\end{aligned}} \tag{37}$$

These define in (33) $p(\underline{x}(t+1)/Z(t), U(t))$

25

THE UPDATE CYCLE:

The Form of

$p\left(\underline{z}(t+1)/\underline{x}(t+1), Z(t), U(t)\right)$

- Measurement equation

$$\underline{z}(t+1) = \underbrace{\underline{C}(t+1)\,\underline{x}(t+1)}_{\text{view this as given}} + \underline{\theta}(t+1) \tag{38}$$

$$\rightarrow p\left(\underline{z}(t+1)/\underline{x}(t+1), Z(t), U(t)\right) = p\left(\underline{z}(t+1)/\underline{x}(t+1)\right) \tag{39}$$

Recall eq (24), Bayes rule:

$$p\left(\underline{x}(t+1)\,|\,Z(t+1), U(t)\right) = \frac{p\left(\underline{z}(t+1)\,|\,\underline{x}(t+1), Z(t), U(t)\right)\,p\left(\underline{x}(t+1)\,|\,Z(t), U(t)\right)}{p\left(\underline{z}(t+1)\,|\,Z(t), U(t)\right)}$$

26

- Since $\underline{\theta}(t+1)$ is gaussian

$$\rightarrow \boxed{p\left(\underline{z}(t+1)/\underline{x}(t+1), Z(t), U(t)\right) \text{ is gaussian}} \tag{40}$$

$$E\left\{\underline{z}(t+1)/\underline{x}(t+1), Z(t), U(t)\right\} = \underline{C}(t+1)\,\underline{x}(t+1) \tag{41}$$

$$\text{cov}\left[\underline{z}(t+1);\ \underline{z}(t+1)/\underline{x}(t+1), Z(t), U(t)\right] = \underline{\Theta}(t+1) \tag{42}$$

27

THE FORM OF $p\left(\underline{z}(t+1)/Z(t), U(t)\right)$

- Measurement equation

$$\underline{z}(t+1) = \underline{C}(t+1)\underline{x}(t+1) + \underline{\theta}(t+1) \tag{43}$$

$\underline{x}(t+1)$, $\underline{\theta}(t+1)$ independent

$p\left(\underline{x}(t+1)\,|\,Z(t), U(t)\right)$ is gaussian (44)

$E\left\{\underline{x}(t+1)\,|\,Z(t), U(t)\right\} = \hat{\underline{x}}(t+1/t)$ (45) one-step predicted mean

$\text{cov}\left[\underline{x}(t+1);\ \underline{x}(t+1)/Z(t), U(t)\right] = \underline{\Sigma}(t+1/t)$ (46) one-step predicted covariance

28

- Also $\underline{\theta}(t+1)$ gaussian

$\Rightarrow$ $\boxed{p(\underline{z}(t+1)/Z(t), U(t)) \text{ gaussian}}$ (47)

$E\{\underline{z}(t+1)/Z(t), U(t)\} =$ (48)

$\underline{C}(t+1)\hat{\underline{x}}(t+1/t)$

$\text{cov}[\underline{z}(t+1); \underline{z}(t+1)/Z(t), U(t)]$

$= \underline{C}(t+1)\, \underline{\Sigma}(t+1/t)\, \underline{C}'(t+1)$

$+ \underline{\Theta}(t+1)$ (49)

29

- Now apply the results of static case to establish that

$\boxed{p(\underline{x}(t+1)/Z(t+1), U(t)) \text{ is gaussian}}$ (50)

- Updated Estimate

$\hat{\underline{x}}(t+1/t+1) \triangleq$ (51)

$E\{\underline{x}(t+1)/Z(t+1), U(t)\}$

- Updated Covariance

$\underline{\Sigma}(t+1/t+1) \triangleq \text{cov}[\underline{x}(t+1); \underline{x}(t+1)/$ (52)

$Z(t+1), U(t)]$

30

CALCULATIONS

$$\boxed{\begin{aligned}\underline{\Sigma}(t+1/t+1) = \underline{\Sigma}(t+1/t) - \underline{\Sigma}(t+1/t)\underline{C}'(t+1).\\ \cdot[\underline{C}(t+1)\,\underline{\Sigma}(t+1/t)\underline{C}'(t+1) + \underline{\Theta}(t+1)]^{-1}\underline{C}(t+1)\,\underline{\Sigma}(t+1/t)\end{aligned}}$$

updated covariance

(53)

$$\boxed{\begin{aligned}\hat{\underline{x}}(t+1/t+1) = \hat{\underline{x}}(t+1/t) + \underline{\Sigma}(t+1/t+1)\underline{C}'(t+1)\,\underline{\Theta}^{-1}(t+1).\\ \cdot[\underline{z}(t+1) - \underline{C}(t+1)\hat{\underline{x}}(t+1/t)]\end{aligned}}$$

updated mean

(54)

31

STOCHASTIC ESTIMATION

The Discrete Time Kalman Filter
Part 2.

32

SUMMARY OF DISCRETE KALMAN FILTER

• State Dynamics:

$$\underline{x}(t+1) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{L}(t)\underline{\xi}(t) \quad (55)$$

• Measurements:

$$\underline{z}(t+1) = \underline{C}(t+1)\underline{x}(t+1) + \underline{\theta}(t+1) \quad (56)$$

33

OFF-LINE CALCULATIONS

• Initialization (t = 0):

$$\underline{\Sigma}(0/0) = \mathrm{cov}\left[\underline{x}(0); \underline{x}(0)\right] \quad (57)$$

• Predict Cycle:

$$\underline{\Sigma}(t+1/t) = \underline{A}(t)\underline{\Sigma}(t/t)\underline{A}'(t) + \underline{L}(t)\underline{\Xi}(t)\underline{L}'(t) \quad (58)$$

• Update Cycle:

$$\underline{\Sigma}(t+1/t+1) = \underline{\Sigma}(t+1/t) - \underline{\Sigma}(t+1/t)\underline{C}'(t+1) \cdot \left[\underline{C}(t+1)\underline{\Sigma}(t+1/t)\underline{C}'(t+1) + \underline{\Theta}(t+1)\right]^{-1} \underline{C}(t+1)\underline{\Sigma}(t+1/t) \quad (59)$$

34

Filter Gain Matrix

$$\underline{H}(t+1) = \underline{\Sigma}(t+1/t+1)\underline{C}'(t+1)\underline{\Theta}^{-1}(t+1) \quad (60)$$

35

ON-LINE CALCULATIONS

- Initialization: $(t = 0)$

$$\hat{\underline{x}}(0/0) = E\{\underline{x}(0)\} \qquad (61)$$

- Predict Cycle:

$$\hat{\underline{x}}(t+1/t) = \underline{A}(t)\hat{\underline{x}}(t/t) + \underline{B}(t)\underline{u}(t) \qquad (62)$$

- Update Cycle

$$\underbrace{\hat{\underline{x}}(t+1/t+1)}_{\text{Updated Estimate}} = \underbrace{\hat{\underline{x}}(t+1/t)}_{\text{Predicted Estimate}} + \underbrace{\underline{H}(t+1)}_{\text{Filter Gain}} \Big\{ \underbrace{\underline{z}(t+1) - \underline{C}(t+1)\hat{\underline{x}}(t+1/t)}_{\text{Residual } \underline{r}(t+1)} \Big\} \qquad (63)$$

RESIDUAL:

$$\underline{r}(t+1) \triangleq \underline{z}(t+1) - \underline{C}(t+1)\hat{\underline{x}}(t+1|t)$$

$$= \underline{C}(t+1)\underline{x}(t+1) + \underline{\theta}(t+1) - \underline{C}(t+1)\hat{\underline{x}}(t+1|t)$$

$$= \underline{C}(t+1)\left[\underline{x}(t+1) - \hat{\underline{x}}(t+1)\right] + \underline{\theta}(t+1)$$

- Residual is zero-mean

$$E\{\underline{r}(t+1)\} = \underline{0}$$

- Residual covariance $\underline{S}(t+1)$

$$\underline{S}(t+1) \triangleq cov\left[\underline{r}(t+1); \underline{r}(t+1)\right]$$

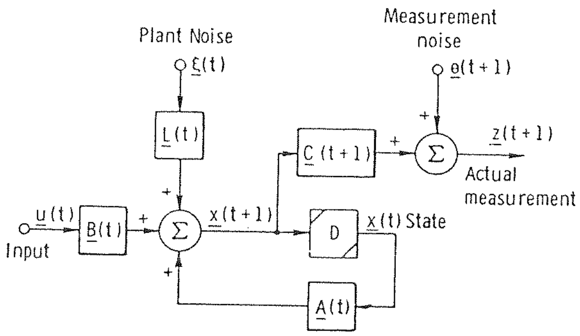
$$= \underline{C}(t+1)\underline{\Sigma}(t+1|t)\underline{C}'(t+1) + \underline{\Theta}(t+1)$$

- Residuals are discrete-time white-noise sequence

$$cov\left[\underline{r}(t); \underline{r}(\tau)\right] = \underline{S}(t)\,\delta_{t\tau}$$

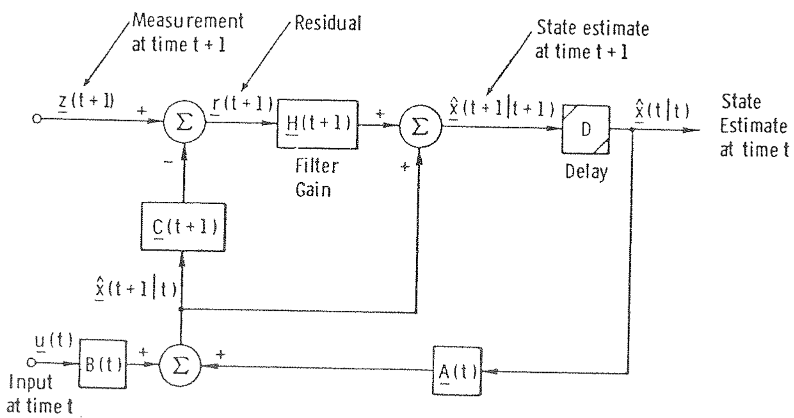
36

STRUCTURE OF SYSTEM
DYNAMICS AND MEASUREMENTS

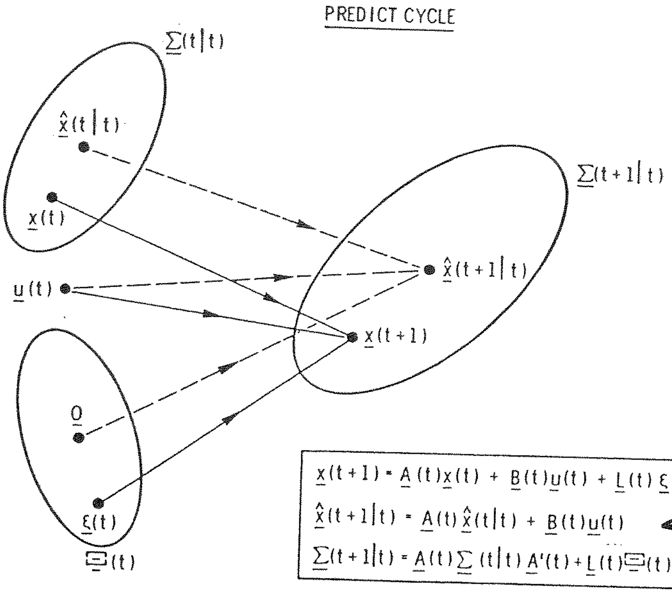


37

STRUCTURE OF DISCRETE-TIME
KALMAN FILTER



38

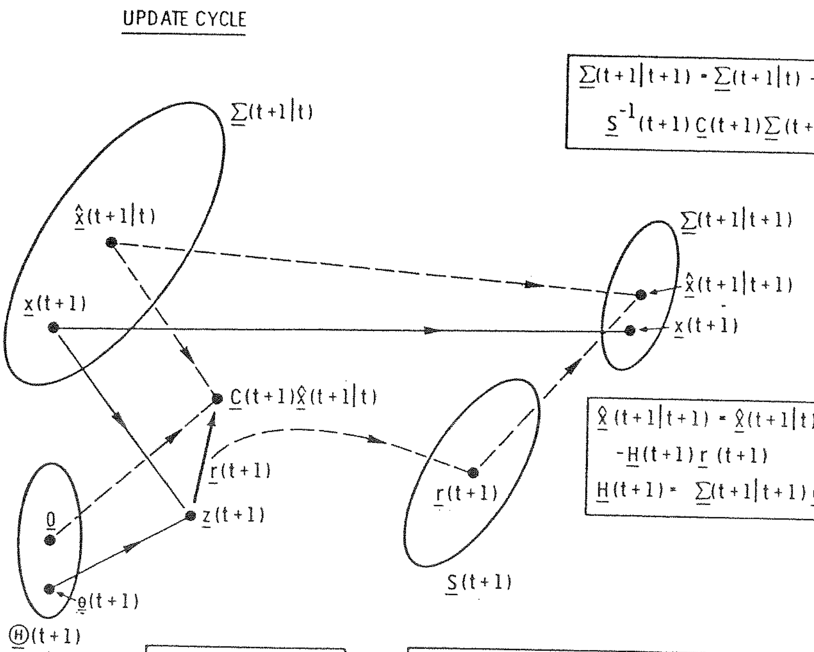


$$x(t+1) = A(t)x(t) + B(t)u(t) + L(t)\xi(t)$$

$$\hat{x}(t+1|t) = A(t)\hat{x}(t|t) + B(t)u(t)$$ ← predicted mean

$$\Sigma(t+1|t) = A(t)\Sigma(t|t)A'(t) + L(t)\Xi(t)L'(t)$$ ← predicted covariance

39



$$\Sigma(t+1|t+1) = \Sigma(t+1|t) - \Sigma(t+1|t)C'(t+1)\cdot S^{-1}(t+1)C(t+1)\Sigma(t+1|t)$$ updated covariance

$$\left.\begin{array}{l}\hat{x}(t+1|t+1) = \hat{x}(t+1|t) \\ \quad -H(t+1)r(t+1)\end{array}\right\}$$ ← updated mean

$$H(t+1) = \Sigma(t+1|t+1)C'(t+1)\Theta^{-1}(t+1)$$ ← KF gain matrix

$$r(t+1) \triangleq z(t+1) - C(t+1)\hat{x}(t+1|t)$$

Residual

$$S(t+1) = C(t+1)\Sigma(t+1|t)C'(t+1) + \Theta(t+1)$$

Residual Covariance.

All covariances independent of deterministic input $u(t)$ and measurements $z(t)$. They can be calculated off-line