Kalman Filter for Continuous-Time Dynamics and Discrete-Time Measurements

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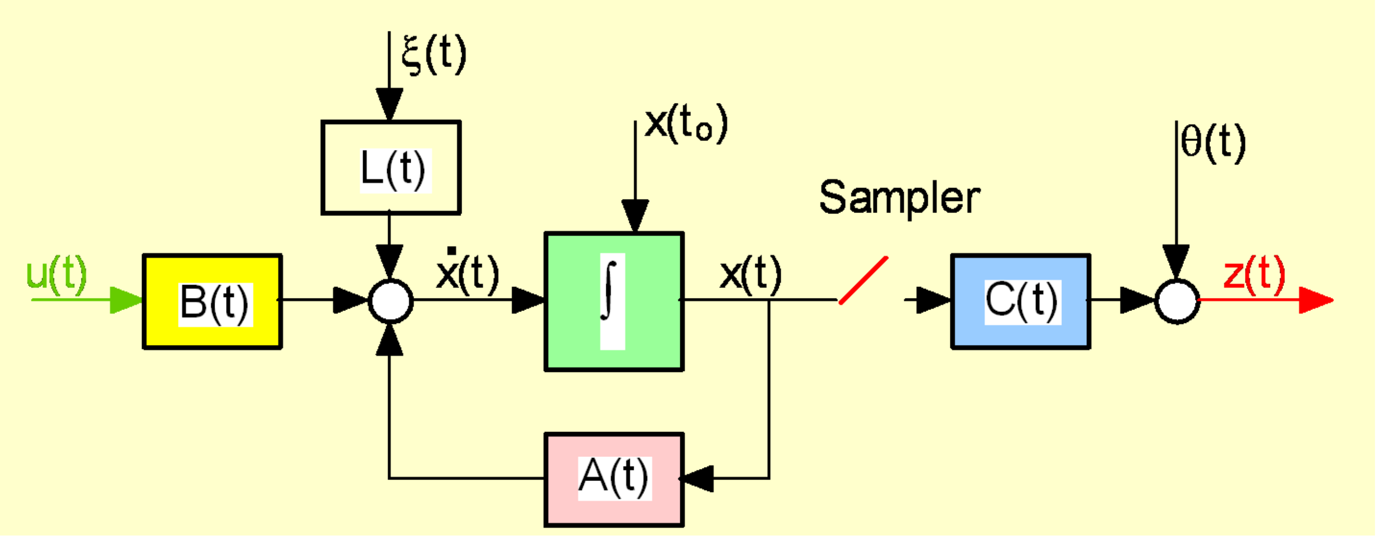
Last Revised: November 19, 2001

Ref. No. KF #8B

Theme

- Summarize Kalman filter equations for the common case of
  - continuous-time plant dynamics
  - discrete-time noisy sensor measurements
- This model is very useful since most physical dynamical systems are naturally modeled by continuous-time stochastic differential equations, but sensors only make measurements at discrete instants of time

Visualization



- The discrete-time measurements are visualized using a SAMPLER that closes at discrete-instants of time

  $t_1, t_2, \ldots, t_k, t_{k+1}, \ldots$

- Thus, the measurements are modeled as

(1) $z(t_k) = C(t_k)x(t_k) + \theta(t_k); \quad k = 1, 2, \ldots$

where $\theta(t_k)$ is discrete-time white noise

Mathematical Modeling

STATE DYNAMICS

- The state satisfies a linear time-varying stochastic differential equation

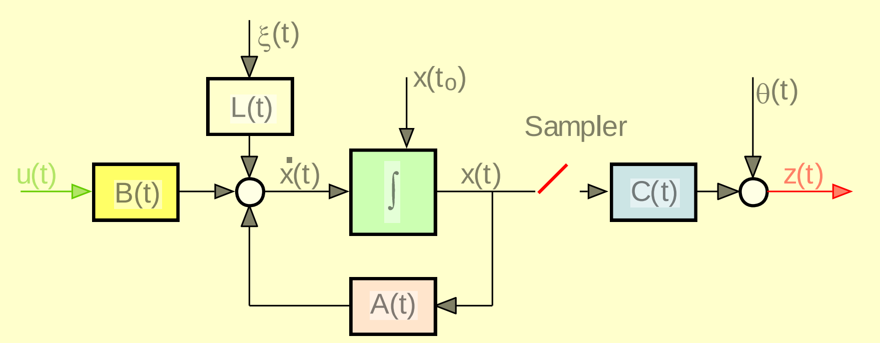
(2) $$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + L(t)\xi(t)$$

where $\xi(t)$ is continuous-time white noise

MEASUREMENTS

- The sensors generate noisy measurements at discrete instants of time in the presence of additive discrete white noise

(3) $$z(t_k) = C(t_k)x(t_k) + \theta(t_k); \quad k = 1, 2, \ldots$$



STATISTICAL INFORMATION

(4) $$E\{x(t_0)\} = \bar{x}_0;\ cov\left[x(t_0);\ x(t_0)\right] = \Sigma_0$$

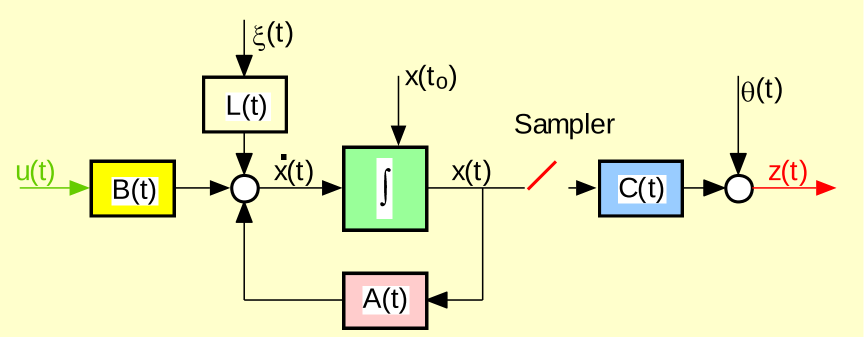
(5) $$E\{\xi(t)\} = 0;\ cov\left[\xi(t);\ \xi(\tau)\right] = \Xi(t)\delta(t-\tau)$$

(6) $$E\{\theta(t_k)\} = 0;\ cov\left[\theta(t_k);\ \theta(t_j)\right] = \Theta(t_k)\delta_{t_k t_j}$$

- All variables are assumed gaussian
- $x(t_0),\ \xi(t),\ \theta(t_k)$ independent

Basic Problem

- We seek continuous- time estimate of the state and of its covariance for all $t \geq t_0$



NOTATION

- $\hat{x}(t \mid t_k)$: predicted state - estimate at time t, $t > t_k$, given the most recent past measurement $z(t_k)$ at time t_k and given the past input time - function $u(\tau)$, $t_0 \leq \tau \leq t$
- $\Sigma(t \mid t_k)$: predicted covariance matrix at time t, $t > t_k$, given the most recent past measurement $z(t_k)$ at time t_k and given the past input time - function $u(\tau)$, $t_0 \leq \tau \leq t$
- $\hat{x}(t_{k+1} \mid t_{k+1})$: updated state - estimate at time t_{k+1}, given the new measurement $z(t_{k+1})$ at time t_{k+1} and given the entire past input time - function $u(\tau)$, $t_0 \leq \tau \leq t_{k+1}$
- $\Sigma(t_{k+1} \mid t_{k+1})$: updated covariance at time t_{k+1}, given the new measurement $z(t_{k+1})$ at time t_{k+1} and given the entire past input time - function $u(\tau)$, $t_0 \leq \tau \leq t_{k+1}$

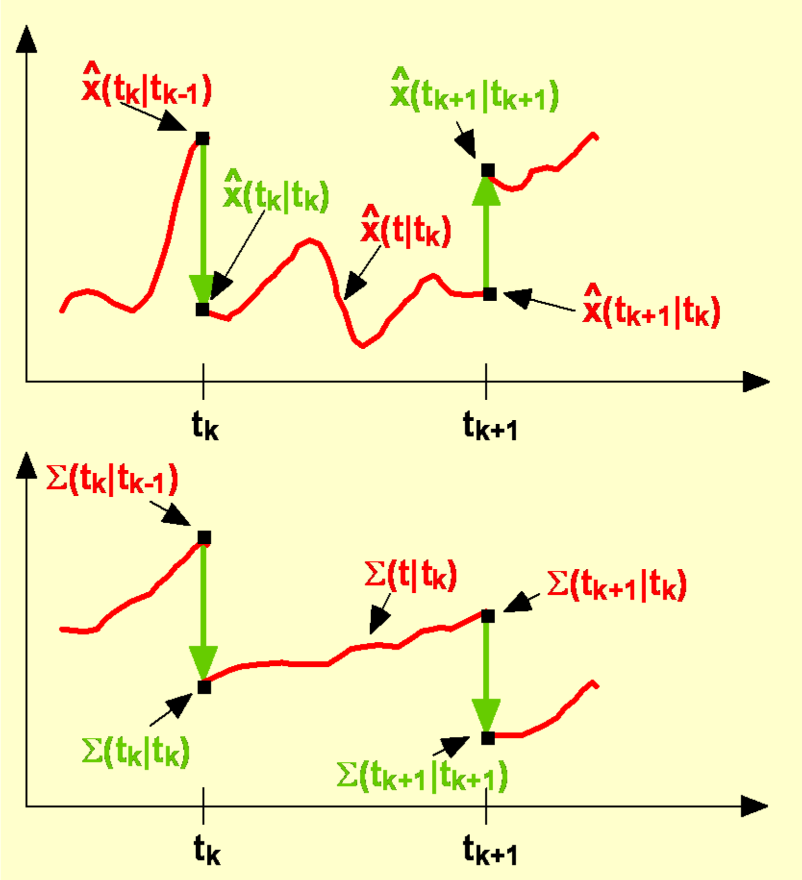
The Basic Idea

PREDICT CYCLE

- Use continuous - time prediction for the state - estimate $\hat{x}(t \mid t_k)$ and covariance $\Sigma(t \mid t_k)$ to obtain $\hat{x}(t_{k+1} \mid t_k)$ and $\Sigma(t_{k+1} \mid t_k)$

UPDATE CYCLE

- Use update cycle formulas for the discrete - time Kalman filter to obtain the state - estimate update $\hat{x}(t_{k+1} \mid t_{k+1})$ and updated covariance matrix $\Sigma(t_{k+1} \mid t_{k+1})$



Filter Equations: Predict Cycle

- Initialization: $\hat{x}(t_0 \mid t_0) = \bar{x}_0; \quad \Sigma(t_0 \mid t_0) = \Sigma_0$
- Predict Cycle: For all $t, \quad t_k < t < t_{k+1},$ the state-estimate $\hat{x}(t \mid t_k)$ is generated by solving the vector differential equation

(7) $\dfrac{d\hat{x}(t \mid t_k)}{dt} = A(t)\hat{x}(t \mid t_k) + B(t)u(t)$ starting at updated $\hat{x}(t_k \mid t_k)$

and the covariance matrix $\Sigma(t \mid t_k)$ is generated by the solution of the matrix Lyapunov differential equation

(8) $\dfrac{d\Sigma(t \mid t_k)}{dt} = A(t)\Sigma(t \mid t_k) + \Sigma(t \mid t_k)A'(t) + L(t)\Xi(t)L'(t)$

starting at updated $\Sigma(t_k \mid t_k)$

- At the next measurement time $t = t_{k+1}$ we calculate from eq. (7) the predicted state $\hat{x}(t_{k+1} \mid t_k),$ and from eq. (8) the predicted covariance $\Sigma(t_{k+1} \mid t_k)$

Filter Equations: Update Cycle

- At time t_{k+1} we obtain the new measurement $z(t_{k+1})$
- The updated state - estimate $\hat{x}(t_{k+1} \mid t_{k+1})$ is obtained as in the discrete - time Kalman filter, i.e.

(9) $$\hat{x}(t_{k+1} \mid t_{k+1}) = \hat{x}(t_{k+1} \mid t_k) + H(t_{k+1})\left[z(t_{k+1}) - C(t_{k+1})\hat{x}(t_{k+1} \mid t_k)\right]$$

- The Kalman filter gain matrix $H(t_{k+1})$ is given by

(10) $$H(t_{k+1}) = \Sigma(t_{k+1} \mid t_{k+1})C'(t_{k+1})\Theta^{-1}(t_{k+1})$$

where the updated covariance matrix $\Sigma(t_{k+1} \mid t_{k+1})$ is given by the discrete - time Kalman filter formula

(11) $$\Sigma(t_{k+1} \mid t_{k+1}) = \Sigma(t_{k+1} \mid t_k) -$$
$$\Sigma(t_{k+1} \mid t_{k+1})C'(t_{k+1})\left[C(t_{k+1})\Sigma(t_{k+1} \mid t_{k+1})C'(t_{k+1}) + \Theta(\ t_{k+1}\ )\right]^{-1} \cdot$$
$$\cdot C(t_{k+1})\Sigma(t_{k+1} \mid t_{k+1})$$

Discussion

- The results presented represent the intuitive blending of the appropriate concepts of discrete-time and continuous-time Kalman filters
- As to be expected, under the gaussian assumptions the conditional probability density of the state, given past measurements and control functions, is gaussian
  - therefore, all state estimates (predicted and updated) represent true conditional means, and
  - all covariance matrices (predicted and updated) represent true conditional covariances