REF. NO. KF #9

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STOCHASTIC ESTIMATION

The Steady-State Kalman-Bucy Filter: Continuous-Time Case

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MOTIVATION

We seek special properties of continuous-time Kalman-Bucy filters where

- state dynamics are linear and time-invariant
- measurement equation is linear and time-invariant
- the noise statistics are stationary

Optimal only in statistical steady-state (effects of initial state gone).

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SUMMARY OF RESULTS

1) As $t \to \infty$, the error covariance matrix $\underline{\Sigma}(t) \to \underline{\Sigma}$ = constant

2) The filter gain matrix $\underline{H}(t) \to \underline{H}$ = constant

3) The Kalman-Bucy filter is linear and time-invariant; it can be constructed using purely analog elements (integrators, constant gains, summers).

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PROBLEM FORMULATION

- State Dynamics (LTI)

$$\boxed{\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) + \underline{L}\,\underline{\xi}(t)} \qquad (1)$$

- Measurement Equation

$$\boxed{\underline{z}(t) = \underline{C}\underline{x}(t) + \underline{\theta}(t)} \qquad (2)$$

- $\underline{A}$, $\underline{B}$, $\underline{L}$, $\underline{C}$ constant known matrices

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INITIAL STATE $\underline{x}(0)$

- $\underline{x}(0)$ is gaussian random vector

$$E\left\{\underline{x}(0)\right\} = \underline{\bar{x}}_0 \tag{3}$$

$$\operatorname{cov}\left[\underline{x}(0); \underline{x}(0)\right] = \underline{\Sigma}_0 = \underline{\Sigma}_0' \geq \underline{0} \tag{4}$$

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PLANT NOISE $\underline{\xi}(t)$

$\underline{\xi}(t)$ is stationary, gaussian, white noise

$$E\left\{\underline{\xi}(t)\right\} = \underline{0} \tag{5}$$

$$\operatorname{cov}\left[\underline{\xi}(t); \underline{\xi}(\tau)\right] = \underline{\Xi}\,\delta(t-\tau) \tag{6}$$

$$\underline{\Xi} = \underline{\Xi}' > \underline{0} \quad \left(\underline{\Xi} = \text{constant matrix}\right) \tag{7}$$

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MEASUREMENT NOISE $\underline{\theta}(t)$

- $\underline{\theta}(t)$ is stationary, gaussian, white noise

$$E\left\{\underline{\theta}(t)\right\} = \underline{0} \tag{8}$$

$$\operatorname{cov}\left[\underline{\theta}(t); \underline{\theta}(\tau)\right] = \underline{\Theta}\,\delta(t-\tau) \tag{9}$$

$$\underline{\Theta} = \underline{\Theta}' > \underline{0} \quad \left(\underline{\Theta} = \text{constant matrix}\right) \tag{10}$$

- $\underline{x}(0)$, $\underline{\xi}(t)$, $\underline{\theta}(\tau)$ mutually independent for all t, τ

Recall:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) + \underline{L}\,\underline{\xi}(t)$$

$$\underline{z}(t) = \underline{C}\,\underline{x}(t) + \underline{\theta}(t)$$

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ADDITIONAL ASSUMPTIONS

- Controllability

$[\underline{A}, \underline{L}]$ is controllable pair * (11)

$\underline{\xi}(t)$ must excite all modes

- Observability

$[\underline{A}, \underline{C}]$ is observable pair ** (12)

all modes must be observed in $\underline{z}(t)$

* $\operatorname{rank}\left[\underline{A} \,\vdots\, \underline{A}\,\underline{L} \,\vdots\, \underline{A}^2\underline{L} \,\vdots\, \cdots \,\vdots\, \underline{A}^{n-1}\underline{L}\right] = n$

** $\operatorname{rank}\left[\underline{C}' \,\vdots\, \underline{A}'^2\underline{C}' \,\vdots\, \cdots \,\vdots\, \underline{A}'^{n-1}\underline{C}'\right] = n$

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TIME VARYING KALMAN-BUCY FILTER

The error covariance matrix $\underline{\Sigma}(t)$ is solution of matrix Riccati differential equation

$$\boxed{\begin{aligned}\frac{d}{dt}\underline{\Sigma}(t) = \underline{A}\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}' \\ - \underline{\Sigma}(t)\underline{C}'\underline{\Theta}^{-1}\underline{C}\,\underline{\Sigma}(t)\end{aligned}} \tag{13}$$

$$\underline{\Sigma}(0) = \text{cov}\left[\underline{x}(0);\ \underline{x}(0)\right] \tag{14}$$

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- The filter gain matrix $\underline{H}(t)$ is

$$\underline{H}(t) = \underline{\Sigma}(t)\underline{C}'\underline{\Theta}^{-1} \tag{15}$$

- The state estimate $\hat{\underline{x}}(t)$ is generated by

$$\boxed{\begin{aligned}\frac{d}{dt}\hat{\underline{x}}(t) = \underline{A}\,\hat{\underline{x}}(t) + \underline{B}\underline{u}(t) \\ + \underline{H}(t)\left[\underline{z}(t) - \underline{C}\hat{\underline{x}}(t)\right]\end{aligned}} \tag{16}$$

$$\hat{\underline{x}}(0) = E\left\{\underline{x}(0)\right\} \tag{17}$$

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ASYMPTOTIC PROPERTIES $t \to \infty$

The error covariance matrix $\underline{\Sigma}(t)$ approaches a constant limiting value $\underline{\Sigma}$

$$\boxed{\lim_{t \to \infty} \underline{\Sigma}(t) = \underline{\Sigma}} \tag{18}$$

and, hence,

$$\lim_{t \to \infty} \underline{H}(t) = \lim_{t \to \infty} \underline{\Sigma}(t)\underline{C}'\underline{\Theta}^{-1} = \underline{H} \tag{19}$$

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The steady state error covariance matrix $\underline{\Sigma}$ is an unique, positive definite, symmetric solution matrix of the algebraic matrix Riccati equation

In MATLAB the "correct" $\underline{\Sigma}$ is computed directly.

$$\boxed{\underline{0} = \underline{A}\underline{\Sigma} + \underline{\Sigma}\underline{A}' + \underline{L}\,\underline{\Xi}\,\underline{L}' - \underline{\Sigma}\underline{C}'\underline{\Theta}^{-1}\underline{C}\,\underline{\Sigma}} \tag{20}$$

many solution matrices possible!

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- The steady-state Kalman-Bucy filter generates an estimate $\hat{\underline{x}}(t)$ by

$$\boxed{\frac{d}{dt}\hat{\underline{x}}(t) = \underline{A}\hat{\underline{x}}(t) + \underline{B}\underline{u}(t) + \underline{H}\left[\underline{z}(t) - \underline{C}\hat{\underline{x}}(t)\right]} \qquad (21)$$

$$\hat{\underline{x}}(0) = E\{\underline{x}(0)\} \qquad (22)$$

NOTE: The filter (21) can be realized with standard analog hardware

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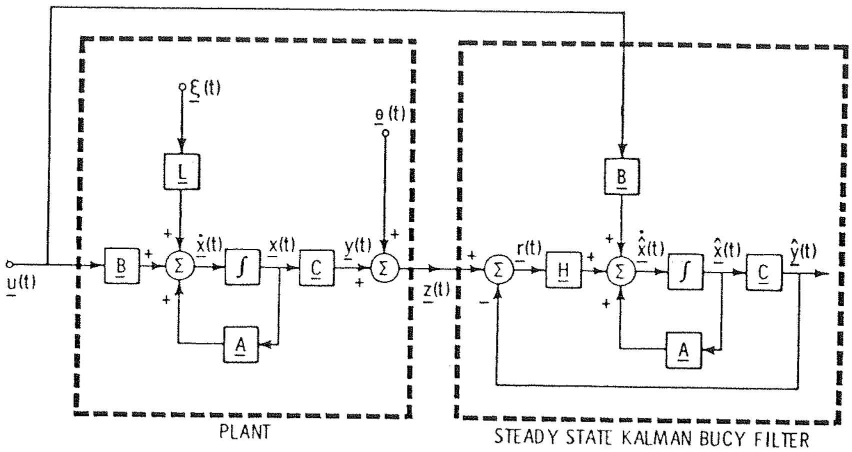

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PROPERTIES

- Alternate form of steady state Kalman-Bucy filter, eq (21)

$$\boxed{\frac{d}{dt}\hat{\underline{x}}(t) = \left[\underline{A}-\underline{H}\underline{C}\right]\hat{\underline{x}}(t) + \underline{B}\underline{u}(t) + \underline{H}\underline{z}(t)} \qquad (23)$$

- Consequence of controllability and observability assumptions.

- Filter "system" matrix $[\underline{A}-\underline{H}\underline{C}]$ is automatically strictly stable, i.e. all of the eigenvalues of $[\underline{A}-\underline{H}\underline{C}]$ have negative real parts

- $\underline{A}$ can be unstable!

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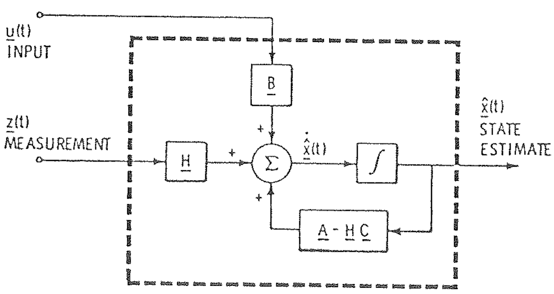

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RESIDUAL PROPERTY

- As $t \to \infty$, the residual or innovations process $\underline{r}(t)$

$$\boxed{\underline{r}(t) \triangleq \underline{z}(t) - \underline{C}\hat{\underline{x}}(t)} \qquad (24)$$

is gaussian, stationary, zero mean continuous white noise

$$\boxed{\text{cov}\left[\underline{r}(t);\ \underline{r}(\tau)\right] = \underline{\Theta}\,\delta(t-\tau)} \qquad (25)$$

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SPECIAL CASE: NO PLANT NOISE

The only uncertainty is due to initial state

- If

$$\underline{\xi}(t) = \underline{0} \quad \left(\text{set } \underline{\Xi} = \underline{0} \text{ or } \underline{L} = \underline{0}\right) \qquad (26)$$

then

$$\dot{\underline{\Sigma}}(t) = \underline{A}\,\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{A}' - \underline{\Sigma}(t)\underline{C}'\underline{\Theta}^{-1}\underline{C}\,\underline{\Sigma}(t) \qquad (27)$$

← Time-varying KF is optimal!

$$\lim_{t \to \infty} \underline{\Sigma}(t) = \bar{\underline{\Sigma}} = \underline{0} \qquad (28)$$

$\dot{\Sigma}$ equation (27) stable, even if $\underline{A}$ is unstable.

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- This implies that we have perfect estimation

$$\lim_{t \to \infty} \hat{\underline{x}}(t) = \underline{x}(t) \qquad (29)$$

Note that

$$\lim_{t \to \infty} \underline{H}(t) = \lim_{t \to \infty} \underline{\Sigma}(t)\underline{C}'\,\underline{\Theta}^{-1} = \bar{\underline{H}} = \underline{0} \qquad (30)$$

i.e. filters runs "open-loop"

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- Hence it makes no sense to have steady state Kalman-Bucy filter if there is no plant noise
- It makes sense to include $\underline{\Xi} > \underline{0}$ (fictitious plant noise) to prevent filter divergence and compensate for plant parameter uncertainties.