Nonlinear Observers with Linearizable Error Dynamics

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Theme

Consider the class of non-autonomous nonlinear systems

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x} &= f(x,u) \\ y &= h(x) \end{cases}$$

where $x(t) \in R^n$ is the vector of system states, $u(t) \in R^m$ are the inputs, and $y(t) \in R^p$ are the system outputs expressed as a column vector and abb. as x, u, and y.

The search for a "very special" property...

Given a nonlinear system,with nonlinear measurements of the state available, find a coordinate transformation that renders the dynamics and the output linear on the new coordinates!!!
(except for a nonlinear output injection term)

Theme

- Challenge for the control problem set at IFAC 1978 (Helsinki) by Roger Brockett to Arthur Krener [1]
- Control problem well understood (during the 80s), see [1, 2] for a survey on the new techniques: feedback linearization, input-output linearization, backstepping , zero dynamics, …
- Harder to be solved for nonlinear observers

Relevant questions:

- Conditions for the existence of such transformation
- Synthesis methods (complexity)
- Robustness relative to unmodelled dynamics...

Krener and Isidori @ 1983

$$z = \varphi(x)$$
$$w = \gamma(y)$$

1) Find $\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases}$ $\rightarrow$ $\begin{cases} \dot{z} = Az + Bu + \alpha(y,u) \\ w = Cz \end{cases}$!

2) Design an observer of the form

$$\dot{\hat{z}} = A\hat{z} + Bu + \alpha(y,u) + K(w - C\hat{z})$$

3)Error dynamics $\tilde{z} = \hat{z} - z$ (assume observability of *(A,C)*)

$$\dot{\tilde{z}} = (A - KC)\tilde{z}$$

- First systematic approach [3] that resorts to a nonlinear state transformation to linearize the original system up to an additional output injection term

Krener and Isidori @ 1983

The proposed solution proposed is composed of three steps (see [1, 3] for details):

1) A set of partial differential equations (PDE) must be solved to find $g(y)$
2) The integrability of conditions for this PDE involve the vanishing of a pseudo-curvature
3) A coordinate transformation $z=f(x)$ can be obtained after a set of PDEs is solved, resorting to conditions on the Lie derivatives of the outputs

"The process is more complicated then feedback linearization and even less likely to be successful..." in [1]

Kazantzis and Kravaris @ 1997

Slitghly different objective:

Given a nonlinear system, with nonlinear measurements of its state available, find a nonlinear state transformation that renders the observer error dynamics linear!!!

(except for a nonlinear output injection term)

Consider the class of autonomous nonlinear systems

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \tag{2}$$

where $f: R^n \rightarrow R^n$, $h: R^n \rightarrow R^m$ are analytic vector fields.

The origin $x = 0$ is an equilibrium point, $f(0) = 0$, and $h(0) = 0$.

Kazantzis and Kravaris @ 1997

Motivated by Luenberger's original ideas on the linear observer design problem, the proposed approach will try to reconstruct a nonlinear invertible function $z = \theta(x)$.

with time derivative that verifies

$$\dot{z} = \frac{\partial \theta}{\partial x}\frac{dx}{dt} = \frac{\partial \theta}{\partial x}\dot{x} = Az - \beta(y)$$

Using the definition of the system (2) and for the intended dynamics, the following PDE must be verified

$$\frac{\partial \theta}{\partial x}(x) f(x) = A\theta(x) - \beta(h(x)) = Az - \beta(y) \qquad (3)$$

Kazantzis and Kravaris @ 1997

Assumption A1 : The Jacobian F of the vector field $f(x)$ evaluated at $x = 0$ has eigenvalues k_i, $i = 1,\dots, n$ with

$$0 \notin ConvexHull\{k_1,\dots,k_n\}$$

Assumption A2 : Denoting the m x n matrix $H = \left[\frac{\partial h_1}{\partial x}^T(0) \ \dots \ \frac{\partial h_m}{\partial x}^T(0)\right]$

it is assumed that the m x n matrix

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \text{ has rank } n.$$

$A2$ essecially states that (3) is locally stable.

Kazantzis and Kravaris @ 1997

Lyapunov's Auxiliary Theorem: Consider the first-order system of quasi-linear differential equations

$$\frac{\partial w}{\partial x}\varphi(x,w)=\psi(x,w) \quad (4)$$

with $\varphi(0,0)=0$, $\psi(0,0)=0$, and $\frac{\partial \varphi}{\partial w}(0,0)=0$, where w is the unknown. Under assumptions ***A1***, ***A2***, and independence of the eigenvalues of $\frac{\partial \varphi}{\partial x}(0,0)$ relative to the ones of $\frac{\partial \psi}{\partial w}(0,0)$, then the above system of PDEs admits a unique analytic solution, in the neighborhood of $x=0$.

The novelty in [4] was the use of this result app to (3) to guarantee the existence and uniqueness of solutions.

Kazantzis and Kravaris @ 1997

Solution for the system of PDEs $$\frac{\partial\theta}{\partial x}(x)f(x) = A\theta(x) - \beta(y)$$

Linear Method : $For \frac{\partial w}{\partial x}\varphi(x,w) = \psi(x,w)$, consider the linear case

$$\varphi(x,w) = Fx$$

$$\psi(x,w) = Aw - BHx$$

with $F, A, B = \frac{\partial\beta}{\partial x}(0)$, and H constant matrices. Then the unique solution of (3) is $w = Tx$, where T is the solution of

$$TF + AT = BH. \tag{5}$$

Unique solution when F and A do not have common eigenvalues.

Kazantzis and Kravaris @ 1997

Theorem: Consider that for the dynamic system (2) ***A1*** and ***A2*** hold and the n - th order dynamic system of the form

$$\dot{z} = Az - \beta(y)$$

where A is Hurwitz, $B = \frac{\partial \beta}{\partial x}(0)$, and (A,B) is controlable.

Then there exists a locally invertible nonlinear map $z = \theta(x)$ that makes the dynamic system above a full order observer.

Why is this method or structure acttractive?...

Kazantzis and Kravaris @ 1997

Theorem: Let $z=\theta(x)$ be an invertible solution of (3). The system

$$\dot{\hat{x}} = f(\hat{x}) - \left[\frac{\partial\theta}{\partial\hat{x}}(\hat{x})\right]^{-1}\left(\beta(y) - \beta(h(\hat{x}))\right) \qquad (3)$$

is an asymptotic full-order observer for (2) such that

$$\frac{d}{dt}(\hat{z}-z) = \frac{d}{dt}\left(\theta(\hat{x})-\theta(x)\right) = A\left(\theta(\hat{x})-\theta(x)\right) = A(\hat{z}-z).$$

Proof (brief) :

$$\frac{d}{dt}\left(\theta(\hat{x})-\theta(x)\right) = \frac{\partial\theta}{\partial\hat{x}}\dot{\hat{x}} - \frac{\partial\theta}{\partial x}\dot{x} = \frac{\partial\theta}{\partial\hat{x}}\left(f(\hat{x}) - \left[\frac{\partial\theta}{\partial\hat{x}}(\hat{x})\right]^{-1}\left(\beta(y)-\beta(h(\hat{x}))\right)\right) - \frac{\partial\theta}{\partial x}f(x) =$$

$$A\theta(\hat{x}) - \beta\left(h(\hat{x})\right) - \left(\beta(y) - \beta\left(h(\hat{x})\right)\right) - A\theta(x) + \beta(y) = A\left(\theta(\hat{x}) - \theta(x)\right)$$

Kazantzis and Kravaris @ 1997

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

Observer design

In original coordinates

$$\dot{\hat{x}} = f(\hat{x}) - \left[\frac{\partial \theta}{\partial \hat{x}}(\hat{x})\right]^{-1} \left(\beta(y) - \beta(h(\hat{x}))\right)$$

$$z = \theta(x)$$

$$\hat{z} = \theta(\hat{x})$$

$$\dot{z} = Az - \beta(y)$$

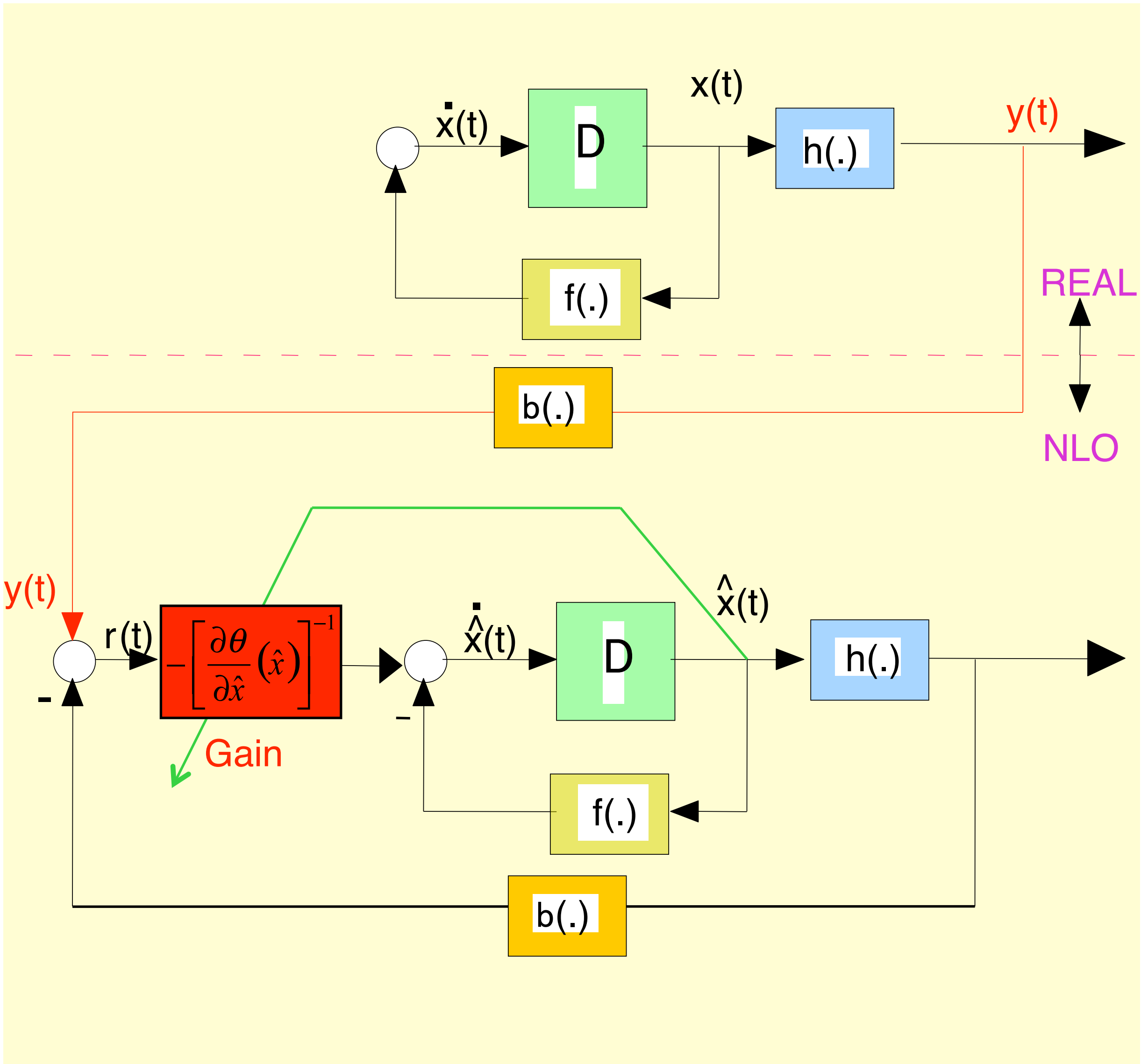
Observer in new coordinates

$$\dot{\hat{z}} = A\hat{z} - \beta(y)$$

$$\dot{\tilde{z}} = A\tilde{z}$$

Linear error dynamics

Visualization of the Filter



Krener and Xiao @ 2002 [5]

Converse Theorem: Consider the class of nonlinear systems

$$\dot{z} = g(z)$$
$$y = h(z)$$

where g and h are continuous vector fields and $g(0) = h(0) = 0$.

If there exists a nonlinear observer $\dot{\hat{z}} = \hat{g}(\hat{z}, y)$ such that the error dynamics $\tilde{z} = z\text{-}\hat{z}$ is linear, i.e. $\dot{\tilde{z}} = A\tilde{z}$, then there exists a continuous vector field $\beta : R^p \to R^n$ such that

$$g(z) = Az - \beta(h(z))$$
$$\hat{g}(\hat{z}) = A\hat{z} - \beta(y)$$

Example I from [5]

3. Examples. As discussed in the introduction, there are distinct advantages to considering *nonlinear output injection* $\beta(y)$. It is desirable that θ be a diffeomorphism over as large a range as possible, for this is the domain of convergence of the observer. Nonlinear output injection can make θ a global diffeomorphism.

To illustrate this, we consider a Duffing oscillator

$$\ddot{x} = x - x^3,$$
$$y = x,$$

which is equivalent to the planar system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix},$$
$$y = x_1.$$

Example I from [5]

This system is trivially transformed into a linear system with output injection (1.2)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} -2y \\ -3y + y^3 \end{bmatrix}$$

by

$$\theta(x) = x,$$
$$\beta(y) = \begin{bmatrix} -2y \\ -3y + y^3 \end{bmatrix}.$$

Notice that β is nonlinear and θ is trivially a global diffeomorphism. The observer (1.4) is

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - \begin{bmatrix} -2y \\ -3y + y^3 \end{bmatrix},$$

and the error dynamics

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

is linear and exponentially stable with poles at $-1 \pm i$.

Example I from [5]

The example is trivial but illustrates two important facts. The first is the advantage of allowing nonlinear β. We could take it to be linear,

$$\beta(y) = \begin{bmatrix} -2 \\ -3 \end{bmatrix} y,$$

and still solve the PDE (1.3) for θ. But the solution might be hard to find, it could have an infinite power series expansion, and it might not be a global diffeomorphism.

The second point is that the Duffing oscillator is truly nonlinear; it has three equilibria and two homoclinic orbits, and the rest of the trajectories are limit cycles. Yet it is possible to build a globally convergent error with linear error dynamics.

Run demo!

Example II from [5]

Next we consider a Van der Pol oscillator,

$$\ddot{x} = -(x^2 - 1)\dot{x} - x,$$
$$y = x,$$

which is equivalent to the planar system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix},$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Now we have

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + x_2 - x_1^2 x_2 \end{bmatrix}, \qquad h(x) = x_1,$$
$$F = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Example II from [5]

We look for a nonlinear coordinate transformation $z = \theta(x)$ such that in the new coordinates z, the system can be described in the form

$$\dot{z} = Az - \beta(y).$$

Let us choose A and β to be

$$A = \begin{bmatrix} b_1 & 1 \\ b_2 - 1 & 1 \end{bmatrix}, \qquad \beta(y) = \begin{bmatrix} b_1 y + \frac{y^3}{3} \\ b_2 y + \frac{y^3}{3} \end{bmatrix},$$

where b_1, b_2 are constants such that $1 + b_1 < 0$, $b_1 - b_2 + 1 > 0$. Clearly, A is stable since $\mathrm{trace}(A) = 1 + b_1 < 0$ and $\det(A) = b_1 - b_2 + 1 > 0$. Moreover $A = F + BH$ with $B = [b_1, b_2]'$. The solution of (1.3) in this case is given by

$$\theta(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^3}{3} \end{bmatrix}.$$

Note that θ is polynomial and *globally invertible* on $\mathbf{R}^2$. This is because we chose a nonlinear β. The resulting observer is again globally convergent with exponentially stable linear error dynamics in $\tilde{z}$ coordinates despite the nonlinearities of the Van der Pol oscillator. See Figure 1.

Example II from [5]

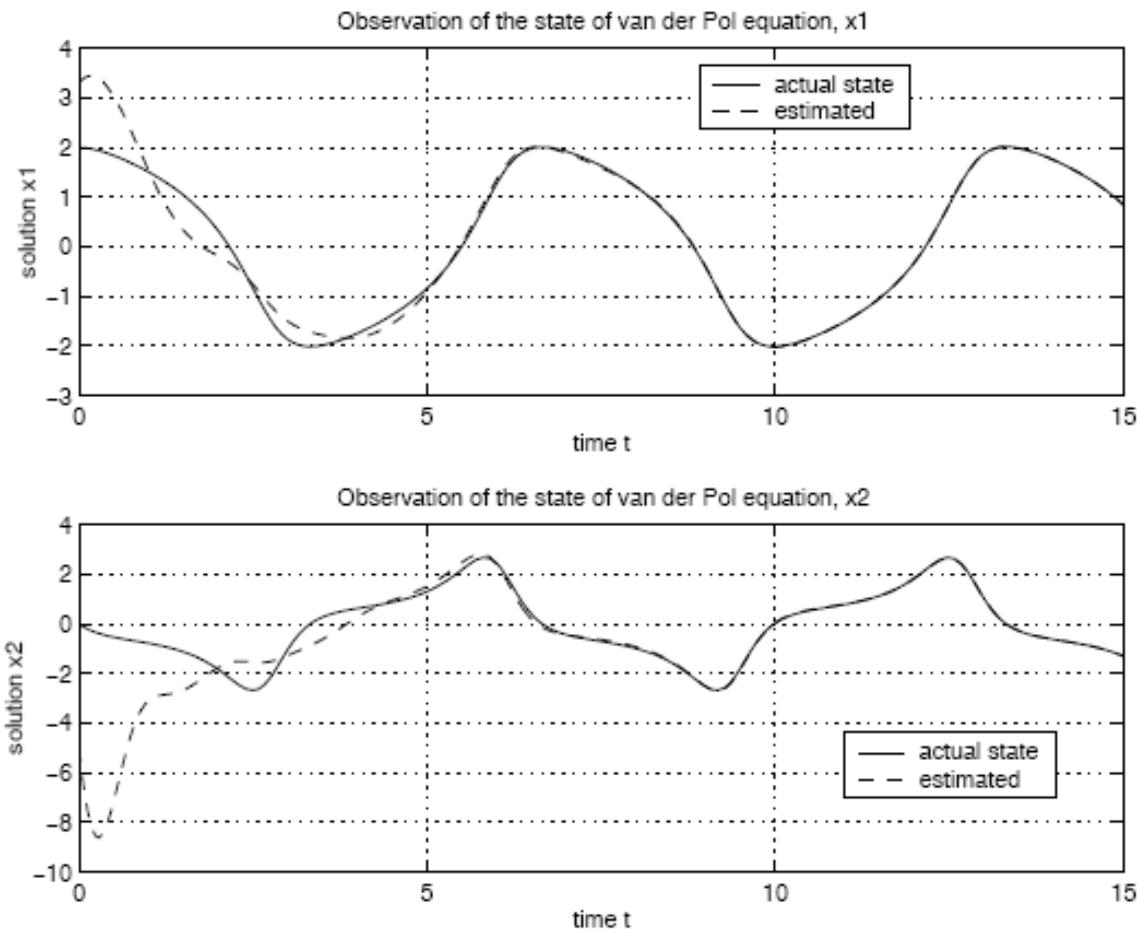


FIG. 1. *Observation of Van der Pol oscillator.*

Both these examples could be treated by the method of Krener and Respondek [8].

Krener and Xiao @ 2002 [5]

Consider the class of non - autonomous nonlinear systems

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

where $f : R^n \to R^n$, $h : R^n \to R^m$ are analytic vector fields.

The origin $x = 0$ is an equilibrium point, $f(0) = 0$, and $h(0) = 0$.

Assume that the following relations are verified

$$\begin{cases} f(x,u) = f_0(x) + f_1(x,u) \\ h(x,u) = h_0(x) + h_1(x,u) \end{cases}$$

where $f_1(x,0) = 0, h_1(x,0) = 0$ and $f_0(0) = h_0(0) = 0.$

Let $F = \dfrac{\partial f_0}{\partial x}(0), H = \dfrac{\partial h_0}{\partial x}(0)$, and $B = \dfrac{\partial \beta}{\partial x}(0)$.

Krener and Xiao @ 2002 [5]

Applying the previous results, under the same technical conditions to the pair f_0, h_0, and for the nonlinear coodinate transformation $z = \varphi(x)$, $i.e$

$$\frac{\partial \varphi}{\partial x}(x) f_0(x) = A\varphi(x) - \beta(h_0(x))$$

requires the solution of the equation

$$TF = TA - BHT.$$

The following nonlinear observer is obtained

$$\dot{\hat{x}} = f(\hat{x}, u) - \left[\frac{\partial \varphi}{\partial \hat{x}}\right]^{-1} \left(\beta(y) - \beta(h(\hat{x}, u))\right)$$

Krener and Xiao @ 2002 [5]

Let $e = \varphi(\hat{x}) - \varphi(x)$. Then e verifies the differential equation

$$\dot{e} = \frac{\partial \varphi}{\partial \hat{x}} f(\hat{x},u) - \left(\beta(y) - \beta(h(\hat{x},u))\right) - \frac{\partial \varphi}{\partial x} f(x,u)$$

$$= \frac{\partial \varphi}{\partial \hat{x}} \left(f_0(\hat{x}) + f_1(\hat{x},u)\right) - \left(\beta(y) - \beta(h(\hat{x},u))\right) - \frac{\partial \varphi}{\partial x} \left(f_0(x) + f_1(x,u)\right)$$

Given the relations verified for the pairs f_0, h_0 in PDE form, i.e

$$\frac{\partial \phi}{\partial x}(x) f_0(x) = A\phi(x) - \beta(h_0(x)), \qquad \frac{\partial \phi}{\partial x}(\hat{x}) f_0(\hat{x}) = A\phi(\hat{x}) - \beta(h_0(\hat{x}))$$

$$\dot{e} = Ae + N(\hat{x},u) - N(x,u)$$

where $N(x_i,u) = \frac{\partial \varphi}{\partial x}(x_i) f_1(x_i,u) + \beta(h(x_i,u)) - \beta(h_0(x_i))$.

Krener and Xiao @ 2002 [5]

If we further assume that f_1 is locally Lipschitz then

$$\left\| N(x_1, u) - N(x_2, u) \right\| \leq L(u) \left\| x_1 - x_2 \right\|$$

A design similar to the ones introduced in the previous lesson is possible, i.e. for A Hurwitz, then for any $Q \geq 0$ then there exist a $P \geq 0$ such that

$$A^T P + PA = -2Q$$

And for the Lyapunov candidate function $V(e) = e^T Pe$ we have that

$$\dot{V}(e) \leq \left(-2\lambda_{\min}(Q) + 2L(u)\lambda_{\max}(P)\right)\left\| e \right\|$$

Hence if

$$\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > L(u)$$

then $e = 0$ is locally asymptotically stable.

Krener and Xiao @ 2002 [5]

- Other methods to solve the PDE could be used [5]

- Design method easier to be accomplished than [3]

- The authors of [4] claim “to be able to do so for all linearly observable, real analytic systems whose spectrum of the linear part lies wholly in the right half complex plane”.

- Krener and Xiao extended the method to arbitrary specta [5] (the Siegel domain) and showed that the sufficient conditions were also necessary.

- Discrete time [6] and state and disturbance estimation design [7] versions became available

References

[1] Mitter S., Baillieul J., and Willems J., "Mathematical Control Theory," 1998.

[2] Kokotovic P. and Arcak M., "Constructive Nonlinear Control: A Historical Perspective," Elsevier, 2000.

[3] Krener A. and Isidori A. , *"Linearization by Output Injection and Nonlinear Observers",* Systems and Control Letters, nº3, pp. 47-52, 1983.

[4] Kazantzis N. and Kravaris C., *"Nonlinear Observer Design using Lyapunov's Auxiliary Theorem,"* CDC, San Diego,1997.

[5] Krener A. and Xiao M., *"Nonlinear Observer Design in the Siegel Domain,"* SIAM Journal of Control and Optimization, v. 41, nº 3, pp. 932-953, 2002.

[6] Xiao M., Kazantzis N., Kravaris C., and Krener A., *"Nonlinear Discrete-time Observer Design with Linearizable Error Dynamics,"* IEEE Trans. Automatic Control, v.48, nº4, pp. 622-626, 2003.

[7] Kravaris C., Sotiropoulos V., Georgiou C., Kazantzis N., Xiao M., and Krener A., *"Nonlinear Observer Design for State and Disturbance Estimation,"* Systems and Control Letters, v. 57, pp. 730-735, 2007.