Nonlinear trajectory tracking control based on adaptive backstepping

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July 7th, 2010

Outline

Trajectory tracking for quadrotors

Quadrotor – underactuated vehicle, nonlinear dynamics

Desired controller properties

Guarantee asymptotic stability of the tracking error at the origin

- with actuation bounded in tracking error
- in the presence of constant force disturbances

Design methodology

- Dynamic model handled in its natural space $\mathrm{SE}(3) = \mathbb{R}^3 \times \mathrm{SO}(3)$
- Nonlinear Lyapunov-based techniques
  - Adaptive backstepping
- Dynamic augmentation of the actuation

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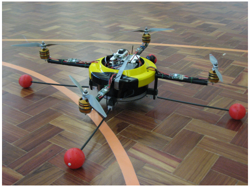
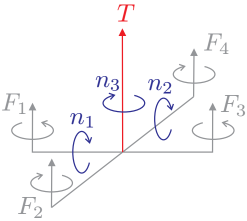
Quadrotor model – Actuation

- 2 pairs of counter-rotating rotors (F_1, F_3) and (F_2, F_4)
- Input transformation

$$T = \sum_{i=1}^{4} F_i$$

$$\mathbf{n} = \begin{bmatrix} \alpha_1(F_1 - F_3) \\ \alpha_2(F_2 - F_4) \\ \alpha_3(F_1 - F_2 + F_3 - F_4) \end{bmatrix}$$





Underactuated vehicle

- One-directional thrust T
- Full torque control $\mathbf{n} = [n_1 \; n_2 \; n_3]^T$
- 1 force + 3 torques vs. 6 DoF

Quadrotor Model – Equations of motion

Kinematics

$$\begin{aligned}\dot{\mathbf{p}} &= R\mathbf{v} \\ \dot{R} &= -S(\boldsymbol{\omega})R\end{aligned}$$

Dynamics

$$\begin{aligned}\dot{\mathbf{v}} &= -S(\boldsymbol{\omega})\mathbf{v} + \frac{1}{m}\mathbf{f} \\ \dot{\boldsymbol{\omega}} &= -\mathbb{J}^{-1}S(\boldsymbol{\omega})\mathbb{J}\boldsymbol{\omega} + \mathbb{J}^{-1}\mathbf{n}\end{aligned}$$

- External force includes thrust and gravitational contributions

$$\mathbf{f} = -T\mathbf{u}_3 + mgR^T\mathbf{u}_3 + R^T\mathbf{b}, \quad \mathbf{u}_3 = [0\ 0\ 1]^T$$

- External force disturbance: $\mathbf{b} \in \mathbb{R}^3$
- Using input transformation $\mathbf{n} = \mathbb{J}\boldsymbol{\tau} + S(\boldsymbol{\omega})\mathbb{J}\boldsymbol{\omega}$, Euler equations are reduced to integrator form

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} = [\tau_1\ \tau_2\ \tau_3]^T$$

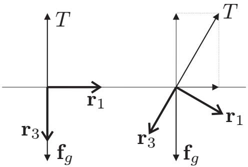
Trajectory tracking – Sketch of problem

Premise Vehicle is underactuated – cannot describe arbitrary trajectories in $\mathrm{SE}(3)$

Solution Enforce position tracking only (define desired position $\mathbf{p}_d(t)$)

Orientation automatically constrained to direct T (desired rotation matrix $R_d = [\mathbf{r}_{1d}\ \mathbf{r}_{2d}\ \mathbf{r}_{3d}]$ satisfies $\mathbf{r}_{3d}T_d = mg\mathbf{u}_3 + \mathbf{b} - m\ddot{\mathbf{p}}_d$)

$$\mathbf{p}_d(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\mathbf{p}_d(t) = \begin{bmatrix} x(t) \\ 0 \\ 0 \end{bmatrix}$$

Trajectory tracking – Problem statement

Assumption Desired trajectory $\mathbf{p}_d(t)$ is a class C^4 function

Control objective

Design a control law for T and $\boldsymbol{\tau}$ such that $\mathbf{p} \to \mathbf{p}_d$ with the largest possible basin of attraction.

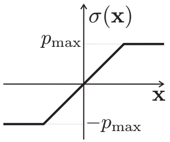
Controller design

- Position error $\mathbf{e}_1 = \mathbf{p} - \mathbf{p}_d$
- Initial candidate Lyapunov function $V_1 = \frac{1}{2}\mathbf{e}_1^T\mathbf{e}_1$
- Go through backstepping procedure until actuation is available

$$\dot{V}_1 = \mathbf{e}_1^T\dot{\mathbf{e}}_1$$

$$\dot{V}_1 = -k_1\mathbf{e}_1^T\sigma(\mathbf{e}_1) + k_1\mathbf{e}_1^T\left(\sigma(\mathbf{e}_1) + \frac{1}{k_1}\dot{\mathbf{e}}_1\right)$$

$$\dot{V}_1 = -W_1(\mathbf{e}_1) + k_1\mathbf{e}_1^T\underbrace{\left(\sigma(\mathbf{e}_1) + \frac{1}{k_1}(R\mathbf{v} - \dot{\mathbf{p}}_d)\right)}_{\mathbf{e}_2}$$



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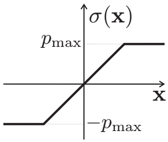

Controller design

- New Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2}\mathbf{e}_2^T\mathbf{e}_2$$
$$\dot{V}_2 = -W_2(\mathbf{e}_1, \mathbf{e}_2) + k_1k_2\mathbf{e}_2^T(\mathbf{e}_2 + \frac{1}{k_1^2k_2}(\frac{1}{m}R\mathbf{f} - \ddot{\mathbf{p}}_d))$$

- With full force control in $\mathbf{f}$, we could obtain
$\dot{V}_2 = -W_2(\mathbf{e}_1, \mathbf{e}_2) < 0$
However $\mathbf{f} = -T\mathbf{u}_3 + mgR^T\mathbf{u}_3 + R^T\mathbf{b}$
and b is unknown!
- Define new error as

$$\mathbf{e}_3 \triangleq \mathbf{e}_2 + \frac{1}{k_1^2k_2}(\frac{1}{m}(-RT\mathbf{u}_3 + mg\mathbf{u}_3 + \widehat{\mathbf{b}}) - \ddot{\mathbf{p}}_d)$$
$$\dot{V}_2 = -W_2(\mathbf{e}_1, \mathbf{e}_2) + k_1k_2\mathbf{e}_2^T\mathbf{e}_3 + \frac{1}{mk_1}\mathbf{e}_2^T\widetilde{\mathbf{b}}$$

and iterate backstepping process twice more

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Controller design – Adaptive backstepping

$$V_4 = \frac{1}{2}\mathbf{e}^T\mathbf{e} \qquad \mathbf{e} = [\mathbf{e}_1^T\ \mathbf{e}_2^T\ \mathbf{e}_3^T\ \mathbf{e}_4^T]^T \in \mathbb{R}^{12}$$

$$\dot{V}_4 = -W_4(\mathbf{e}) + \mathbf{e}_4^T\left(\mathbf{h}(\mathbf{e}, T, \dot{T}, R, \boldsymbol{\omega}, \mathbf{p}_d^{(4)}) + RM(T)\bar{\mathbf{u}} + \frac{1}{k_1^2 k_2}\dot{\hat{\mathbf{b}}}\right) + \frac{1}{k_1^2 k_2}\mathbf{e}_3^T\dot{\hat{\mathbf{b}}} + \frac{1}{k_1}\tilde{\mathbf{b}}^T(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4))$$

$$\bar{\mathbf{u}} = \left[\ddot{T} \quad \tau_1 \quad \tau_2\right]^T,\ M(T) = \frac{c_1}{m}\begin{bmatrix} 0 & 0 & -T \\ 0 & T & 0 \\ 1 & 0 & 0 \end{bmatrix}$$, nonsingular if $T \neq 0$

Add estimation error to the Lyapunov function

$$V_4 = \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2k_1}\tilde{\mathbf{b}}^T\Gamma^{-1}\tilde{\mathbf{b}} \qquad \mathbf{e} = [\mathbf{e}_1^T\ \mathbf{e}_2^T\ \mathbf{e}_3^T\ \mathbf{e}_4^T]^T \in \mathbb{R}^{12}$$

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Controller design – Closing the loop

Feedback laws

- Estimator update law: $\dot{\hat{\mathbf{b}}} = \Gamma(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$
- Control law: $\bar{\mathbf{u}} = -M(T)^{-1}R^T \left(\mathbf{h}(.) + \frac{1}{k_1^2 k_2}\dot{\hat{\mathbf{b}}}\right)$

- Combines torque actuation $[\tau_1 \ \tau_2]$ with the 2nd order derivative of the thrust $\ddot{T}$
- τ_3 plays no part in providing trajectory tracking

Closed-loop system

$$\dot{\mathbf{e}} = -K_b\mathbf{e} + [0 \ I_3 \ I_3 \ I_3]^T \frac{1}{k_1}\tilde{\mathbf{b}}$$

$$\dot{\tilde{\mathbf{b}}} = -\Gamma[0 \ I_3 \ I_3 \ I_3]\mathbf{e}$$

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Controller design – Stability analysis

Small position error ($\|\mathbf{e}_1\| < p_{\max}$)

Asymptotic stability of the error system origin is proven by Barbalat's Lemma

Large position error ($\|\mathbf{e}_1\| > p_{\max}$)

Using an auxiliar Lyapunov function

$$V_\sigma = \frac{1}{2}\sum_{i=2}^{4} \mathbf{e}_i^T \mathbf{e}_i + \frac{1}{2}\tilde{\mathbf{b}}^T \Gamma^{-1} \tilde{\mathbf{b}}$$

it can be proven that there is a time T such that for all $t > T$

$$\|\mathbf{e}_1(t)\| < p_{\max}.$$

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Controller design – Exploring the extra DoF

Control law for trajectory tracking leaves $\tau_3 = \dot{\omega}_3$ free.

1st approach

Apply $\tau_3 = -k_3\omega_3$ to stabilize angular velocity ω_3 at zero.

2nd approach

Enforce zero sideslip angle $\Leftrightarrow v_y = [0\ 1\ 0]\mathbf{v} \rightarrow 0$

1. Consider $R_d = [\mathbf{r}_{1d}\ \mathbf{r}_{2d}\ \mathbf{r}_{3d}] \in \mathrm{SO}(3)$

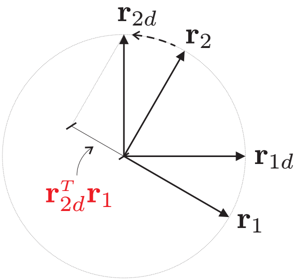
   Thrust direction $\mathbf{r}_{3d}$ already prescribed by trajectory tracking requirements.
2. Define $\mathbf{r}_{2d}$ so that $\mathbf{r}_2 \rightarrow \mathbf{r}_{2d} \Leftrightarrow \mathbf{r}_2^T\mathbf{r}_{2d} \rightarrow 1 \Rightarrow v_y \rightarrow 0$

Controller design – Exploring the extra DoF

2nd approach

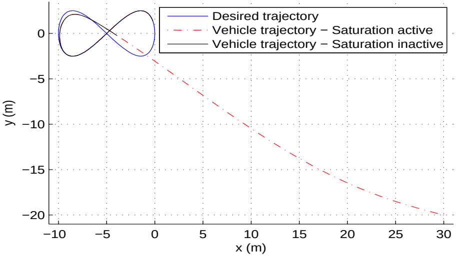
3. Apply PD-like control law

$$\tau_3 = -l_2(\omega_3 - \omega_{3d} + l_1 \mathbf{r}_{2d}^T \mathbf{r}_1) + \dot{\omega}_{3d} - l_1 \frac{d}{dt}(\mathbf{r}_{2d}^T \mathbf{r}_1)$$



$-l_2 l_1 \mathbf{r}_{2d}^T \mathbf{r}_1$ opposes growth in angular distance between $\mathbf{r}_2$ and $\mathbf{r}_{2d}$

Simulation results

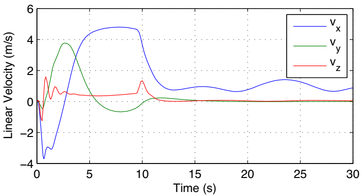


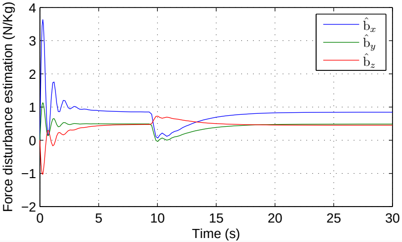
- Track an eight-shaped trajectory

$$\mathbf{p}_d(t) = \begin{bmatrix} 5\cos(0.2t) - 5 \\ 2.5\sin(0.4t) \\ -10 \end{bmatrix}$$

- Saturation active for initial position
- Constant lateral wind disturbance of 4.76 N
- 5 % uncertainty in gravitational acceleration

Simulation results





- Bounded velocity rapidly acquired
- Convergence to desired velocity when saturation is no longer active
- v_y converges to zero

- $\widehat{\mathbf{b}}$ converges to $\mathbf{b}$
- b_z captures uncertainty in g
- b_x, b_y captures wind disturbance

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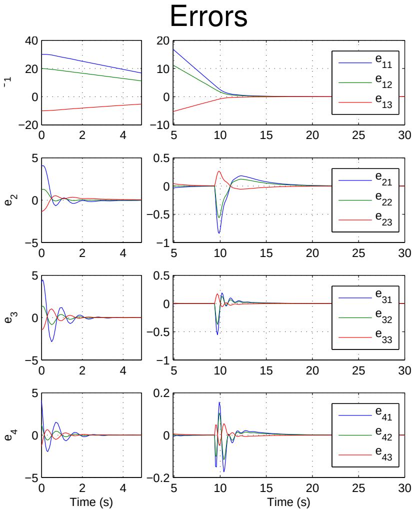

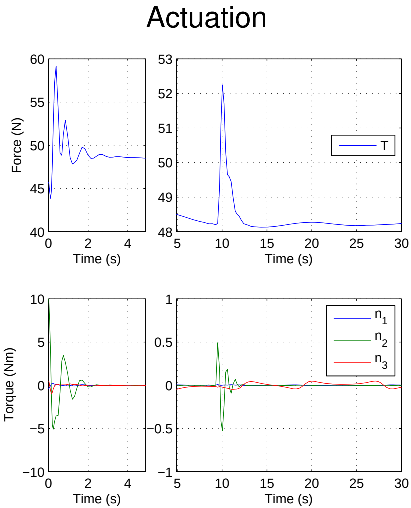

Conclusions

Trajectory tracking control of quadrotors

Solution for underactuated vehicles based on dynamic augmentation of the actuation

- Resorts to
  - Nonlinear systems theory
  - Lyapunov-based stability analysis
  - Adaptive backstepping techniques
- Properties
  - Asymptotic stability in the presence of constant force disturbances
  - Bounded influence of the position error in force and torque actuation

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