Nonlinear trajectory tracking control of a quadrotor vehicle

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Abstract— **This paper addresses the problem of steering a quadrotor vehicle along a time-dependent trajectory. The problem is formulated so as to take into account force disturbances acting on the vehicle and enforce bounds on the actuation. The proposed solution consists of a nonlinear adaptive state feedback controller for thrust and torque actuation that *i)* guarantees asymptotic stability of the closed-loop system in the presence of constant force disturbances and *ii)* ensures that the actuation does not grow unbounded as a function of the position errors. Simulation results are presented to assess the performance and robustness of the proposed controller.**

I. INTRODUCTION

Motion control of underactuated vehicles is an active topic of research, which raises new and challenging problems when measured against motion control of its fully actuated counterpart. The quadrotor, in particular, is a typical example of an underactuated vehicle, ideally suited for the development and test of new control strategies due to its simplicity and maneuverability. In recent years, several approaches to the problem of controlling these rotorcraft have been proposed, among which we highlight the Lyapunov-based backstepping method.

Backstepping is a well known technique extensively used for control of nonlinear systems. For example, it has been applied to helicopter systems for trajectory tracking in [1] and [2] and also for tracking of parallel linear visual features in [3]. In general, the backstepping technique is not applicable to underactuated systems. However, as shown in [4], a simplified model commonly adopted for both quadrotors and helicopters is feedback linearizable by dynamic augmentation of the thrust actuation, and hence stabilizable by means of backstepping.

Several methodologies can be combined with backstepping to attain desirable characteristics of a control law, such as robustness to external disturbances and actuation boundedness. The use of integral control to achieve zero steady-state error or equivalently rejection of constant disturbances in a closed-loop regulation system is standard in control literature and can be combined with the backstepping technique as discussed in [5]. The control methodology known as adaptive backstepping [6] relies on an estimator to achieve the disturbance rejection effect of integral control.

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Important works in the literature on bounded control are presented in [7], [8], and more recently in [9]. An application of these techniques to helicopter control can be found in [10], wherein a controller for horizontal stabilization and vertical reference tracking is proposed. Saturated controls are used to guarantee actuation boundedness and adaptive control techniques are employed for reference tracking and to increase the controller robustness to disturbances.

In this paper, we address the problem of trajectory tracking for quadrotors, using a backstepping procedure that builds on the dynamic augmentation principle presented in [4]. The desired trajectory is specified by a sufficiently smooth time-parameterized position vector. The desired attitude of the vehicle is not prescribed since attitude convergence (up to a rotation about the body z axis) is naturally accomplished by solving the position tracking problem. For controller design, the attitude is handled in its natural space, SO(3), as a rotation matrix. This avoids the introduction of artifacts related only to the parameterization, as is the case of singularities with Euler angles and multiple coverings with the quaternion representation [11].

After deriving the initial control law based on unbounded position errors, a straightforward approach is followed to impose bounds on the actuation. We show that applying the same control law while using a saturation function on the position error that is fed back continues to stabilize the system. Robustness to external constant disturbances is accomplished through adaptive backstepping. These disturbances can be used to represent both exogenous inputs such as constant wind and model uncertainties such as gravitational acceleration mismatches.

The quadrotor's z-axis symmetry leads to a control law with a rotational degree of freedom that can be exploited to control the vehicle's heading as convenient and independently from the trajectory tracking control law. Towards that end, we propose two different heading control solutions. The first simply guarantees that the angular velocity along the body z-axis converges to zero, whereas the second provides convergence of the sideslip angle to zero.

This paper is structured as follows. Section II introduces the quadrotor model. The problem and control objective are stated in Section III. Controller design is described in Section IV, including the necessary steps to ensure disturbance rejection and bounded actuation. Section V proposes strategies for exploring the heading degree of freedom. Simulation results illustrating the performance of the proposed control laws are presented in Section VI and Section VII summarizes the contents of the paper.

II. Quadrotor Model

We model the quadrotor vehicle as a rigid body that is actuated in force and torque and possibly subject to a constant force disturbance, which can model both constant wind disturbances and gravitational constant mismatches. Consider a fixed inertial frame $\{I\}$ and a body frame $\{B\}$ attached to the vehicle's center of mass. The configuration of $\{B\}$ with respect to $\{I\}$ can be viewed as an element of the special euclidean group, $(R, \mathbf{p}) = ({}^I_B R, {}^I\mathbf{p}_B) \in \mathrm{SE}(3) \triangleq \mathrm{SO}(3) \times \mathbb{R}^3$. The kinematic and dynamic equations of motion for the rigid body can be written as

$$\dot{R} = RS(\boldsymbol{\omega}) \tag{1}$$

$$\dot{\mathbf{p}} = R\mathbf{v} \tag{2}$$

$$\dot{\boldsymbol{\omega}} = -\mathbb{J}^{-1}S(\boldsymbol{\omega})\mathbb{J}\boldsymbol{\omega} + \mathbb{J}^{-1}\mathbf{n} \tag{3}$$

$$\dot{\mathbf{v}} = -S(\boldsymbol{\omega})\mathbf{v} + \frac{1}{m}\mathbf{f} + \frac{1}{m}R^T\mathbf{b} \tag{4}$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$ and $\mathbf{v} \in \mathbb{R}^3$ denote the angular and linear velocities expressed in $\{B\}$, m and $\mathbb{J} \in \mathbb{R}^{3\times3}$ represent the quadrotor's mass and moment of inertia, $\mathbf{f}$ and $\mathbf{n} \in \mathbb{R}^3$ denote the external force and torque expressed in $\{B\}$, and $\mathbf{b} \in \mathbb{R}^3$ is the external force disturbance expressed in $\{I\}$. The map $S(.)$ yields a skew symmetric matrix that verifies $S(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}$, for $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$.

Adopting the standard simplified model for quadrotors [4], [10], such that torques can be generated in any direction and the actuation force is always aligned with the body z axis, the external force can be written as

$$\mathbf{f} = -T\mathbf{u}_3 + mgR^T\mathbf{u}_3 \tag{5}$$

where $\mathbf{u}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, $T \in \mathbb{R}$ is the sum of the thrust contributions from the four rotors and g is the gravitational acceleration. With full torque control, the Euler equations of motion (3) can be reduce to the integrator form $\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3]^T$, using the input transformation

$$\mathbf{n} = \mathbb{J}\boldsymbol{\tau} + S(\boldsymbol{\omega})\mathbb{J}\boldsymbol{\omega}. \tag{6}$$

The quadrotor is thus an underactuated vehicle, with more degrees of freedom than actuation variables. In this particular case, only a scalar force actuation T is available, which must be complemented by the torque actuation vector $\mathbf{n}$ to control the translational motion of the vehicle.

III. Problem Formulation

Let the desired trajectory be described by a function of time $\mathbf{p}_d(t) \in \mathbb{R}^3$, which is assumed to be of class C^4. In what follows, time dependence will be omitted to lighten notation. The control objective consists of designing a control law for the quadrotor inputs T and $\mathbf{n}$, which ensures the convergence of the position $\mathbf{p}$ to $\mathbf{p}_d$ with the largest possible basin of attraction.

Due to the underactuated nature of the vehicle, the desired attitude cannot be arbitrarily selected. From (4) and (5), we can verify that the equilibrium for trajectory tracking satisfies

$$R_dT_d\mathbf{u}_3 = mg\mathbf{u}_3 + \mathbf{b} - m\ddot{\mathbf{p}}_d. \tag{7}$$

Consequently, the desired rotation matrix R_d is automatically prescribed up to a rotation about the body z axis ($R_dR_z(\psi)T_d\mathbf{u}_3 = R_dT_d\mathbf{u}_3$), which is in agreement with the fact that we are only constraining three degree of freedoms when four control inputs are available. The symmetry exhibited by the quadrotor vehicle dictates the nature of this particular degree of freedom, given that rotations about the body z axis play no part in directing the thrust vector.

We start by considering the tracking problem assuming that the external force $\mathbf{b}$ is known and unbounded actuation is available. The solution to this problem is then extended to provide bounded actuation and finally to account for external force disturbances.

IV. Controller design

Let the position error vector $\mathbf{e}_1$ be given by

$$\mathbf{e}_1 \triangleq \mathbf{p} - \mathbf{p}_d. \tag{8}$$

For tracking to be attained, the controller must drive this error to zero and we therefore define the first tentative Lyapunov function as the squared norm of the error $\mathbf{e}_1$

$$V_1 \triangleq \frac{1}{2}\mathbf{e}_1^T\mathbf{e}_1. \tag{9}$$

Computing the time derivative V_1, we can write

$$\dot{V}_1 = -W_1(\mathbf{e}_1) + k_1\mathbf{e}_1^T(\mathbf{e}_1 + \frac{1}{k_1}\dot{\mathbf{e}}_1) \tag{10}$$

where $\dot{\mathbf{e}}_1 = R\mathbf{v} - \dot{\mathbf{p}}_d$, $W_1(\mathbf{e}_1) = k_1\mathbf{e}_1^T\mathbf{e}_1$, and $k_1 > 0$. Similarly to W_1, $W_i(\cdot)$, $i \in \{2, 3, 4\}$ is used in the sequel to denote a quadratic function of its arguments that can be made positive definite by appropriately selecting a set of positive coefficients k_j, $j \in \{1, \ldots, i\}$. Applying the standard backstepping procedure, we define a second error and Lyapunov function and compute its time derivatives to obtain

$$\mathbf{e}_2 \triangleq \mathbf{e}_1 + \frac{1}{k_1}\dot{\mathbf{e}}_1 \tag{11}$$

$$V_2 \triangleq \frac{1}{2}\sum_{i=1}^{2}\mathbf{e}_i^T\mathbf{e}_i \tag{12}$$

$$\dot{V}_2 = -W_2(\mathbf{e}_1, \mathbf{e}_2) + k_1k_2\mathbf{e}_2^T\left(\mathbf{e}_2 + \tfrac{1}{k_1^2k_2}\ddot{\mathbf{e}}_1\right) \tag{13}$$

where $\ddot{\mathbf{e}}_1 = \frac{1}{m}(R\mathbf{f} + \mathbf{b}) - \ddot{\mathbf{p}}_d$.

At this point, if full force control were available, a feedback law for $\mathbf{f}$ could be applied to yield $\dot{V}_2 = -W_2(\mathbf{e}_1, \mathbf{e}_2)$ and thus solve the tracking problem. In the present case, rotation motion is required to align the force vector in a suitable direction. Two courses of action can be taken with regard to the thrust actuation. One can first set a control law for thrust and then follow the backstepping process, or, one can work through the backstepping process using a generic thrust until a control law that combines torque actuation and a higher order derivative of the thrust can be defined. The first approach leads to an aggressive time-scale separation of the system dynamics, as observed for the control law proposed in [10]. In this paper, we follow the latter approach, which is also used in [3] and [2].

We apply the backstepping procedure twice more to obtain the errors

$$\mathbf{e}_3 \triangleq \mathbf{e}_2 + \tfrac{1}{k_1^2 k_2}\ddot{\mathbf{e}}_1 \tag{14}$$

$$\mathbf{e}_4 \triangleq \mathbf{e}_3 + c_1 \mathbf{e}_1^{(3)} \tag{15}$$

and associated Lyapunov functions $V_3 \triangleq \frac{1}{2}\sum_{i=1}^{3} \mathbf{e}_i^T \mathbf{e}_i$ and $V_4 \triangleq \frac{1}{2}\sum_{i=1}^{4} \mathbf{e}_i^T \mathbf{e}_i$. The constant c_1 is given by $c_1 = \frac{1}{k_1^2 k_2 (k_1 k_2 + k_3)}$ and $\mathbf{e}_1^{(3)}$ denotes the third-order time derivative of $\mathbf{e}_1$. Defining the complete error state vector

$$\mathbf{e} \triangleq [\mathbf{e}_1^T\, \mathbf{e}_2^T\, \mathbf{e}_3^T\, \mathbf{e}_4^T]^T \tag{16}$$

we can write

$$V_4 = \frac{1}{2}\mathbf{e}^T\mathbf{e} \tag{17}$$

$$\dot{V}_4 = -W_4(\mathbf{e}) + \mathbf{e}_4^T \left(\mathbf{h}(\mathbf{e}, T, \dot{T}, R, \boldsymbol{\omega}, \mathbf{p}_d^{(4)}) + RM(T)\bar{\mathbf{u}}\right) \tag{18}$$

where

$$\mathbf{h}(\mathbf{e}, T, \dot{T}, R, \boldsymbol{\omega}, \mathbf{p}_d^{(4)}) = -\tfrac{c_1}{m} RS(\boldsymbol{\omega})(S(\boldsymbol{\omega})T\mathbf{u}_3 + 2\dot{T}\mathbf{u}_3) - c_1 \mathbf{p}_d^{(4)} + k_1 k_2(\mathbf{e}_3 - \mathbf{e}_2) + (k_1 k_2 + k_3 + k_4)\mathbf{e}_4 \tag{19}$$

and the matrix M and the vector of inputs $\bar{\mathbf{u}}$ are given by

$$M(T) = \tfrac{c_1}{m}\begin{bmatrix} 0 & 0 & -T \\ 0 & T & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{20}$$

and

$$\bar{\mathbf{u}} = \begin{bmatrix} \ddot{T} & \tau_1 & \tau_2 \end{bmatrix}^T \tag{21}$$

respectively. From the definition of M in (20) and $\dot{V}_4$ in (18), it is easy to observe that as long as the thrust T remains positive, we can define a control law for $\bar{\mathbf{u}}$ that guarantees convergence of all the errors to zero. As an initial controller, we consider the following feedback law

$$\bar{\mathbf{u}} = -M^{-1}(T) R^T \mathbf{h}(\mathbf{e}, T, \dot{T}, R, \boldsymbol{\omega}, \mathbf{p}_d^{(4)}) \tag{22}$$

to obtain the closed-loop error system

$$\dot{\mathbf{e}} = -K\mathbf{e} \tag{23}$$

where $K \in \mathbb{R}^{12\times 12}$ depends on the coefficients k_i.

Lemma 1: Consider the closed-loop system (23), which results from applying the controller (22) and arbitrary τ_3 to the quadrotor system described by (1)-(6) and expressing it in error coordinates. For an appropriate choice of coefficients k_i, (23) has an asymptotically stable equilibrium point at the origin.

Proof: Consider V_4 defined in (17) as a candidate Lyapunov function for the closed-loop error system. Substituting (22) in (18) yields

$$\dot{V}_4 = -W_4(\mathbf{e}) = -k_1 \mathbf{e}_1^T \mathbf{e}_1 - k_1(k_2 - 1)\mathbf{e}_2^T \mathbf{e}_2 - k_3 \mathbf{e}_3^T \mathbf{e}_3 - k_4 \mathbf{e}_4^T \mathbf{e}_4 + k_1(\mathbf{e}_3 + \mathbf{e}_4)^T(\mathbf{e}_2 - \mathbf{e}_1). \tag{24}$$

There is a suitable choice of coefficients k_i that makes $\dot{V}_4$ a negative definite function. Using standard Lyapunov arguments, we have that the origin of (23) is asymptotically stable. ∎

A. Bounding the position error

To enforce bounds on the actuation, different questions need to be addressed. First, the reference trajectories should be such that tracking can be achieved while keeping the actuators within their limits of operation. For that purpose, we not only assume that $\mathbf{p}_d(t)$ is a class C^4 function, but also impose bounds on its derivatives. Second, the control law depends linearly on the errors $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$. This is a critical limitation mainly because all these errors depend on the position error $\mathbf{e}_1$, which can be arbitrarily large as opposed to velocities and accelerations that are automatically bounded by the operation limits of the vehicle.

To overcome this difficulty, we consider the same control law as in (22) but redefine the error vector $\mathbf{e} = [\mathbf{e}_1^T\, \mathbf{e}_2^T\, \mathbf{e}_3^T\, \mathbf{e}_4^T]^T$ such that $\mathbf{e}_1$ remains unchanged whereas $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$ become functions of a bounded version of $\mathbf{e}_1$. More specifically, the velocity error $\mathbf{e}_2$ is redefined as

$$\mathbf{e}_2 \triangleq \sigma(\mathbf{e}_1) + \tfrac{1}{k_1}\dot{\mathbf{e}}_1 \tag{25}$$

where σ is the saturation function given by

$$\sigma(\mathbf{x}) \triangleq \begin{cases} \mathbf{x} & \text{if } \|\mathbf{x}\| \le p_{\max} \\ p_{\max}\frac{\mathbf{x}}{\|\mathbf{x}\|} & \text{if } \|\mathbf{x}\| > p_{\max} \end{cases} \tag{26}$$

and this change is propagated to $\mathbf{e}_3$ and $\mathbf{e}_4$ according to (14) and (15), respectively.

To show that the proposed controller continues to be a stabilizing one, we start by considering the function $V_4 = \frac{1}{2}\mathbf{e}^T\mathbf{e}$ as in the previous case and recompute its derivative given the new definition of $\mathbf{e}$. For small position errors $\|\mathbf{e}_p\| < p_{\max}$, both the Lyapunov function and controls remain unchanged and asymptotic stability can thus be proven. For large errors, it can be shown that

$$\dot{V}_4 \le -k_1\|\mathbf{e}_1\|(p_{\max} - \|\mathbf{e}_2\|) - W_5(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4). \tag{27}$$

Once again, a suitable choice of coefficients, ensures that W_5 is positive definite, yielding a negative definite derivative for V_4 when $\|\mathbf{e}_2\| < p_{\max}$. From this analysis, we can only conclude that the system converges when the initial conditions satisfy $\|\mathbf{e}(0)\| < p_{\max}$. A less conservative result can be obtained by showing that $\mathbf{e}_2$ is ultimately bounded by $p_{\max}$. The following theorem shows that this is in fact the case.

Theorem 2: Let the quadrotor kinematics and dynamics be described by (1)-(6), and consider the transformation to error coordinates $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ given by (8), (25), (14), (15), respectively. For an appropriate choice of coefficients k_i, the closed-loop system that results from applying the controller (22) and arbitrary τ_3 to the error system has an asymptotically stable equilibrium point at the origin.

Proof: Consider the Lyapunov function $V_\sigma \triangleq \frac{1}{2}\sum_{i=2}^{4}\mathbf{e}_i^T\mathbf{e}_i$. Straightforward computations show that, for $\|\mathbf{e}_1\| \le p_{\max}$, the following upper bound for $\dot{V}_\sigma$ holds

$$\dot{V}_\sigma \le -k_1((k_2-1)\|\mathbf{e}_2\| - p_{\max})\|\mathbf{e}_2\| + k_1(\mathbf{e}_3 + \mathbf{e}_4)^T\mathbf{e}_2 - (k_3\|\mathbf{e}_3\| - k_1 p_{\max})\|\mathbf{e}_3\| - (k_4\|\mathbf{e}_4\| - k_1 p_{\max})\|\mathbf{e}_4\| \tag{28}$$

whereas for $\|\mathbf{e}_1\| > p_{\max}$, $\dot{V}_\sigma$ is given by

$$\dot{V}_\sigma = -k_1 k_2 \mathbf{e}_2^T \mathbf{e}_2 - k_3 \mathbf{e}_3^T \mathbf{e}_3 - k_4 \mathbf{e}_4^T \mathbf{e}_4 + (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^T \tfrac{1}{p_{\max}\|\mathbf{e}_1\|} S(\sigma(\mathbf{e}_1))^2 \mathbf{e}_2. \quad (29)$$

For $\|[\mathbf{e}_2^T, \mathbf{e}_3^T, \mathbf{e}_4^T]\| > p_{\max}$, $\dot{V}_\sigma$ can be made negative definite through a thoughtful choice of coefficients k_i. We can thus conclude that these errors are ultimately bounded. In particular, there exists a finite time T such that $\|\mathbf{e}_2(t)\| < p_{\max}$ for all $t > T$. Once $\|\mathbf{e}_2(t)\| < p_{\max}$, we can use (27) to verify that $\dot{V}_4$ becomes negative definite and conclude that the origin of error system is asymptotically stable. ∎

B. Unknown constant disturbances

We now explore the situation where the disturbance $\mathbf{b}$ is not exactly known and must be estimated. Throughout this section the disturbance estimate normalized by the mass m is denoted by $\widehat{\mathbf{b}}$ and the estimation error is given by $\widetilde{\mathbf{b}} = \frac{1}{m}\mathbf{b} - \widehat{\mathbf{b}}$.

The external disturbances only affect the rigid body system through (4). As such, the backstepping process is in all similar to the one presented in the previous section up to the definition of $\mathbf{e}_3$ given in (14). Since the exact value of $\mathbf{b}$ is no longer available, we redefine $\mathbf{e}_3$ to depend on the estimate $\widehat{\mathbf{b}}$ instead of on $\mathbf{b}$, so that

$$\mathbf{e}_3 \triangleq \mathbf{e}_2 + \tfrac{1}{k_1^2 k_2}(\tfrac{1}{m}R\mathbf{f} + \widehat{\mathbf{b}} - \ddot{\mathbf{p}}_d) = \mathbf{e}_2 + \tfrac{1}{k_1^2 k_2}(\ddot{\mathbf{e}}_1 - \widetilde{\mathbf{b}}). \quad (30)$$

Keeping the definition for $\mathbf{e}_4$ given in (15) and adding a quadratic term in $\widetilde{\mathbf{b}}$ to V_4, we have

$$V_4 \triangleq \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2k_1}\widetilde{\mathbf{b}}^T \Gamma^{-1} \widetilde{\mathbf{b}} \quad (31)$$

$$\dot{V}_4 = -W_4(\mathbf{e}) + \mathbf{e}_4^T\left(\mathbf{h}(.) + RM(T)\bar{\mathbf{u}} + \tfrac{1}{k_1^2 k_2}\dot{\widehat{\mathbf{b}}}\right) + \tfrac{1}{k_1^2 k_2}\mathbf{e}_3^T \dot{\widehat{\mathbf{b}}} + \tfrac{1}{k_1}\widetilde{\mathbf{b}}^T\left(-\Gamma^{-1}\dot{\widehat{\mathbf{b}}} + (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)\right) \quad (32)$$

where Γ is a positive definite matrix and the matrix M, the vector of controls $\bar{\mathbf{u}}$ and the vector $\mathbf{h}$ are defined in (20), (21), and (19), respectively.

Given the expression for $\dot{V}_4$, we define the estimator update law

$$\dot{\widehat{\mathbf{b}}} = \Gamma(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) \quad (33)$$

and the control law

$$\bar{\mathbf{u}} = -M(T)^{-1}R^T\left(\mathbf{h}(\mathbf{e}, T, \dot{T}, R, \boldsymbol{\omega}, p_d^{(4)}) + \tfrac{1}{k_1^2 k_2}\dot{\widehat{\mathbf{b}}}\right) \quad (34)$$

leading to the Lyapunov function derivative

$$\dot{V}_4 = -k_1\mathbf{e}_1^T\mathbf{e}_1 - k_1(k_2 - 1)\mathbf{e}_2^T\mathbf{e}_2 - k_3\mathbf{e}_3^T\mathbf{e}_3 - k_4\mathbf{e}_4^T\mathbf{e}_4 + k_1(\mathbf{e}_3 + \mathbf{e}_4)^T(\mathbf{e}_2 - \mathbf{e}_1) + \tfrac{1}{k_1^2 k_2}\mathbf{e}_3^T\Gamma(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) \quad (35)$$

and the closed-loop system

$$\dot{\mathbf{e}} = -K_b\mathbf{e} + [0\ I_3\ I_3\ I_3]^T \tfrac{1}{k_1}\widetilde{\mathbf{b}} \quad (36)$$

$$\dot{\widetilde{\mathbf{b}}} = -\Gamma[0\ I_3\ I_3\ I_3]\mathbf{e}. \quad (37)$$

We can now state the following result.

Lemma 3: For an appropriate choice of coefficients k_i, the closed-loop system (36)-(37) has an asymptotically stable equilibrium point at the origin.

Proof: There is a set of coefficients k_i such that $\dot{V}_4$ is a negative semi-definite function, which implies that the origin is stable. Applying LaSalle's invariance principle, we have that $(\mathbf{e}, \widetilde{\mathbf{b}})$ converges to the largest invariant set where $\mathbf{e} = 0$. From the definition of the error coordinates, it follows immediately that this set contains only the origin $\mathbf{e} = 0$ and $\widetilde{\mathbf{b}} = 0$. ∎

Redefining the velocity error $\mathbf{e}_2$ to depend on the bounded position error $\sigma(\mathbf{e}_1)$ as in (25), we can follow the same steps as in Section IV-A to analyze the stability of the closed-loop system.

Theorem 4: Let the quadrotor kinematics and dynamics be described by (1)-(6) and consider the transformation to the error coordinates $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ given by (8), (25), (30), (15), respectively. For an appropriate choice of coefficients k_i and arbitrary torque τ_3, the closed-loop system (36)-(37) that results from applying the controller (34) and the update law (33) has an asymptotically stable equilibrium point at the origin.

Proof: Using $V_\sigma = \frac{1}{2}\sum_{i=2}^{4}\mathbf{e}_i^T\mathbf{e}_i + \frac{1}{2k_1}\widetilde{\mathbf{b}}^T\Gamma^{-1}\widetilde{\mathbf{b}}$ as a Lyapunov function and noting that its derivative is negative semi-definite for all $\|[\mathbf{e}_2^T\ \mathbf{e}_3^T\ \mathbf{e}_4^T]\| \leq p_{\max}$ leads to the result that there exists a finite time T such that $\|\mathbf{e}_2(t)\| < p_{\max}$, for $t > T$. Once $\|\mathbf{e}_2(t)\| < p_{\max}$ we can use the Lyapunov function (31) and note that its derivative is negative semi-definite to prove stability of the closed-loop system and convergence of $\mathbf{e}$ and $\widetilde{\mathbf{b}}$ to zero. ∎

The proposed control law (22) can only be applied if the thrust force never reaches zero. A conservative estimate of the initial states for which the thrust never crosses the origin can be obtained using (30) together with the bounds for the errors and estimation derived from the negative semi-definiteness of (35). In short, we can obtain the following lower bound for $|T(t)|$

$$|T(t)| \geq mg - \|\mathbf{b}\| - m\left\|\mathbf{p}_d^{(3)}(t)\right\| - m(\sqrt{2\rho_{\max}(\Gamma)} + 2\sqrt{2}k_1^2k_2)V_4^{1/2}(0). \quad (38)$$

If the initial conditions and desired trajectories are such that the lower bound for $|T(t)|$ is positive, then the thrust $T(t)$ that results from applying the proposed control law is guaranteed to take only positive values.

V. Exploring the extra degree of freedom

As specified in (7), if tracking of the desired trajectory is accomplished then the rotation matrix is automatically prescribed up to a rotation about the body z axis. This is corroborated by the control law (34) obtained in the previous section, which leaves the torque $\tau_3 = \dot{\omega}_3$ free. We can thus use this extra degree of freedom to meet an additional heading control objective.

As a first approach, we can simply enforce the convergence of ω_3 to zero, applying for example the control law $\tau_3 = -k\omega_3$. A more involved goal that is still attainable is to ensure that the vehicle tracks the reference trajectory with no sideslip angle or equivalently with zero velocity component along the body y axis. Combining (7), which describes the trajectory tracking equilibrium, with the additional constraint

$$[0\ 1\ 0]^T \mathbf{v}_d = v_{yd} = 0 \tag{39}$$

we can obtain the following expression for the desired rotation matrix

$$R_d = \left[-\frac{S(\mathbf{r}_{3d})^2 \dot{\mathbf{p}}_d}{\|S(\mathbf{r}_{3d})\dot{\mathbf{p}}_d\|} \quad \frac{S(\mathbf{r}_{3d})\dot{\mathbf{p}}_d}{\|S(\mathbf{r}_{3d})\dot{\mathbf{p}}_d\|} \quad \mathbf{r}_{3d} \right] \tag{40}$$

whose third column is given by

$$\mathbf{r}_{3d} = \frac{mg\mathbf{u}_3 + \mathbf{b} - m\ddot{\mathbf{p}}_d}{\|mg\mathbf{u}_3 + \mathbf{b} - m\ddot{\mathbf{p}}_d\|}. \tag{41}$$

Clearly, (40) is not well-defined for $S(\mathbf{r}_{3d})\dot{\mathbf{p}}_d = 0$. When this is the case, any pair ($\mathbf{r}_{1d}$,$\mathbf{r}_{2d}$) such that $R_d = [\mathbf{r}_{1d}\ \mathbf{r}_{2d}\ \mathbf{r}_{3d}] \in$ SO(3) yields $v_{yd} = 0$. Since the control law for tracking already ensures that $\mathbf{r}_3$ converges to $\mathbf{r}_{3d}$, to drive v_y to zero, we need only have convergence of $\mathbf{r}_2$ to $\mathbf{r}_{2d}$ or equivalently of $\mathbf{r}_{2d}^T\mathbf{r}_2$ to one.

Using (40), straightforward but rather lengthy computations provide expressions for both ω_{3d} and its time derivative $\dot{\omega}_{3d}$. Gathering all these elements, we arrive at the following PD-like control law for τ_3

$$\tau_3 = -l_2(\omega_3 - \omega_{3d} + l_1\mathbf{r}_{2d}^T\mathbf{r}_1) + \dot{\omega}_{3d} - \frac{d}{dt}(\mathbf{r}_{2d}^T\mathbf{r}_1) \tag{42}$$

with $l_1 > 0$ and $l_2 > 0$. Using the Lyapunov function $W = (\omega_3 - \omega_{3d} + l_1\mathbf{r}_{2d}^T\mathbf{r}_1)^2$ and computing its time derivative along the system's trajectories, we can immediately conclude that $\dot{W} < 0$ and so ω_3 converges to $\omega_{3d} - l_1\mathbf{r}_{2d}^T\mathbf{r}_1$. Clearly this is not enough to guarantee that $\mathbf{r}_2$ is approaching $\mathbf{r}_{2d}$, however (42) gives some indication of providing this convergence since the term $-l_1\mathbf{r}_{2d}^T\mathbf{r}_1$ opposes growth in the angular distance between $\mathbf{r}_2$ and $\mathbf{r}_{2d}$. More formally, we can define the states

$$\xi \triangleq \omega_3 - \omega_{3d} + l_1\mathbf{r}_{2d}^T\mathbf{r}_1 \in \mathbb{R} \tag{43}$$

$$\eta \triangleq 1 - \mathbf{r}_{2d}^T\mathbf{r}_2 \in [0,\ 2] \tag{44}$$

and verify that, at the tracking equilibrium given by $\mathbf{e} = 0$ and $\xi = 0$, the dynamic system for η is described by

$$\dot{\eta} = -(2-\eta)\eta \tag{45}$$

and the origin of (45) is asymptotically stable. If we consider the closed-loop quadrotor system that results from applying the control laws (22) and (42) and express it in error coordinates with state $(\mathbf{e}, \xi, \eta)$, (45) can be thought of as the quadrotor zero dynamics. We can therefore conclude that the overall closed-loop system has an asymptotically stable equilibrium point at the origin, given that $\mathbf{e} = 0$ and $\xi = 0$ are asymptotically stable and so is the origin of the zero dynamics. Determining the respective region of attraction requires further investigation and is left for future work.

As in Section IV-B, assuming that unknown disturbances are present, we can substitute $\mathbf{b}$ by $m\widehat{\mathbf{b}}$ in (41) and compute the corresponding estimates for R_d, ω_{3d}, and $\dot{\omega}_{3d}$. Following the same line of reasoning, it can be shown that the stability result for the full closed-loop system (now with the extra state $\widetilde{\mathbf{b}}$) continues to hold.

VI. Simulation Results

In this section we present the results obtained from a simulation run using the proposed controller. At the initial configuration, the quadrotor's orientation is such that the thrust counteracts the estimated effect of gravity and the distance to the target position is such that the saturation (26) is active. To assess the robustness of the controller, an external constant lateral wind disturbance of magnitude 4.76 N is considered along with a 5% uncertainty in the gravitational acceleration g. The reference trajectory is an eight-shaped path defined by

$$\mathbf{p}_d(t) = \begin{bmatrix} 5\cos(0.2t) - 5 \\ 2.5\sin(0.4t) \\ -10 \end{bmatrix}$$

The vehicle is initially placed at $\mathbf{p} = [30, -20, -20]^T$, with orientation $R = I_3$ and zero velocity. The vehicle mass is $m = 4.76$ kg and the control law coefficients are $k_1 = 0.7$, $k_2 = 4$, $k_3 = 3$, $k_4 = 4$, $\Gamma = 2I_3$, $p_{\max} = 5$, and $l_1 = l_2 = 1$. Fig. 1 presents a view of the desired trajectory and of the trajectory described by the vehicle.

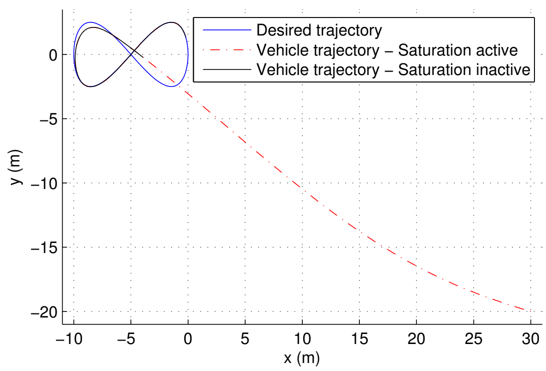

Fig. 1. 2-D view of the vehicle trajectory

As shown in Fig. 2, the quadrotor acquires rapidly a bounded velocity, moving in the direction of the desired position. When the distance to the target is such that the saturation (26) is no longer active, the velocities converge to the desired velocities and trajectory tracking is attained. Notice also that the velocity component v_y converges to zero as prescribed by the heading control law (42).

In Fig. 3, we can observe that after a quick initial transient the estimate of the constant disturbances $\widehat{\mathbf{b}}$ converges to the correct value $\mathbf{b}$. After around 10 s of simulation, a small transient can be observed, which is caused by the deactivation of the saturation in the control law. The gravitational acceleration uncertainty is captured by $\mathbf{b}_z$ while the lateral wind disturbance is captured by $\mathbf{b}_x$ and $\mathbf{b}_y$.

In Fig. 4, we can see that the only significant error after the initial transient is the position error $\mathbf{e}_1$. Additionally, a small increase on the other errors $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ occurs at the moment when the saturation ceases to be active due

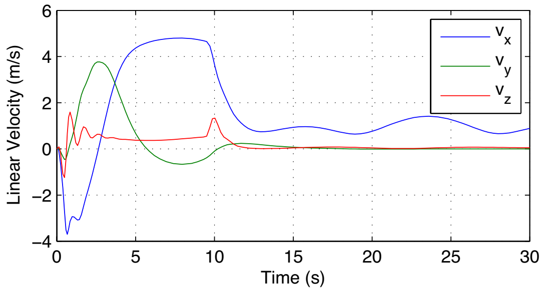

Fig. 2. Linear velocities

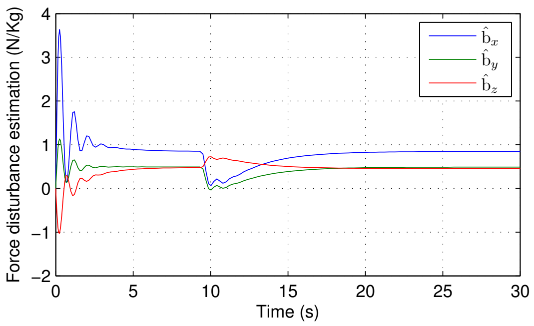

Fig. 3. Force disturbance estimates

to the continuous but nonsmooth reduction of the desired velocity. All actuation signals are bounded and do not vary significantly from their nominal values during the simulation, as shown in Fig. 5.

VII. Conclusions

This paper presented a state feedback solution to the problem of stabilizing an underactuated quadrotor vehicle along a predefined trajectory using bounded thrust and torque actuations in the presence of constant force disturbances. A Lyapunov function for the system was derived using backstepping techniques and an adaptive estimator was introduced so as to compensate for the force disturbance. The bounds on the actuation result from the boundedness of the Lyapunov function and depend on the initial configuration errors.

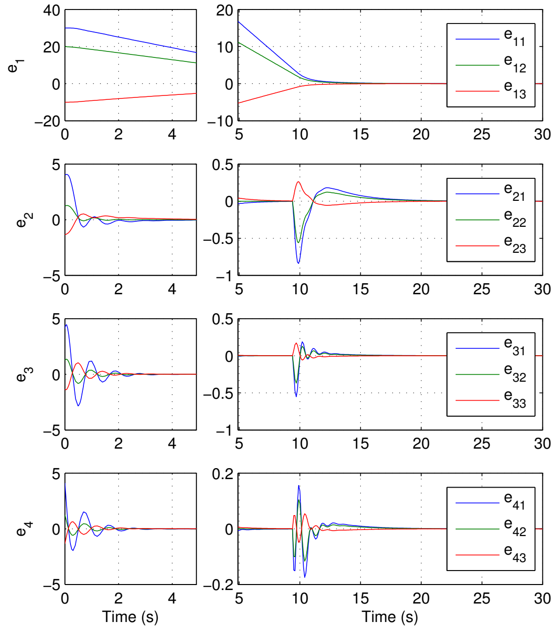

Fig. 4. Trajectory tracking errors

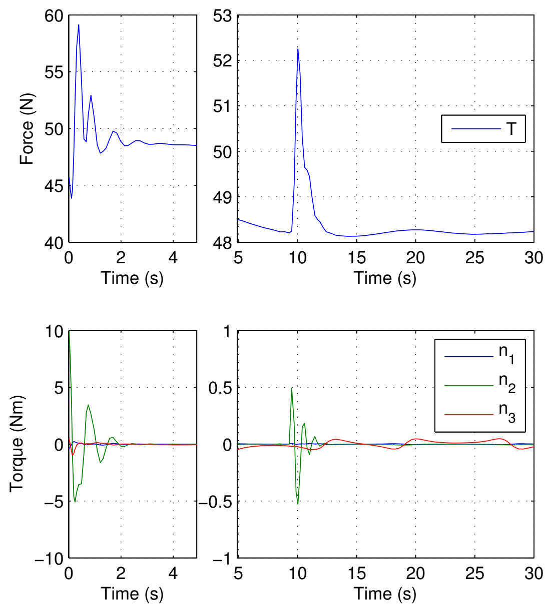

Fig. 5. Actuation

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