DETECTION, ESTIMATION, AND FILTERING COURSE

Kalman filtering with intermittent heavy tailed observations

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Outline

- Motivation
- Problem definition
- Bounds for error convergence
- Adding heavy tails to the picture
- Simulation results
- Conclusions

Motivation

- Wide range of applications of wireless sensor networks:
  - environmental and industrial monitoring, military surveillance, object tracking
- Observed data may be used to estimate the state of a controlled system– estimate later used for control
  - In control– critical that data arrives in time
- In large multi-hop networks, unreliable wireless channels:
  - loss & delay
  - significant delay– from the aspect of controls– same as loss
  - How to perform estimation when some of the observations are missing?

Approach to the problem

- Start with standard Kalman filter
- Model arrival of observations as random Bernoulli variable
- State and error covariance now random variables
- What are statistical convergence properties of the expected estimation error covariance?
- Existence of critical value
- Upper and lower bounds on the expected value of the state error covariance

Problem definition

- Start with discrete-time linear system with Gaussian noise

$$x_{t+1} = Ax_t + w_t$$
$$y_t = Cx_t + v_t$$

$x_t \in \Re^n$ state vector
$y_t \in \Re^m$ output vector
$w_t \in \Re^p$ plant noise
$v_t \in \Re^m$ measurement noise
independent Gaussian - zero mean and covariance Q and R
initial state x_0 -zero mean Gaussian, covariance Σ_0

- Kalman filter is an optimal estimator

Problem definition

- Arrival of the observation at time t :
  - binary random variable γ_t with probability distribution $p_\gamma \quad (1) = \lambda_t$
- Measurement noise v_t defined:

$$p\left(v_t \mid \gamma_t\right) = \begin{cases} \mathcal{N}\left(0, R\right), & \gamma_t = 1 \\ \mathcal{N}\left(0, \sigma^2 I\right), & \gamma_t = 0 \end{cases}$$

  - The absence of observations corresponds to the case of $\sigma \rightarrow \infty$

- Modified Kalman equations:

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} \\ P_{t+1|t} &= AP_{t|t}A' + Q \\ \hat{x}_{t+1|t+1} &= \hat{x}_{t+1|t} + \gamma_{t+1}K_{t+1}\left(y_{t+1} - C\hat{x}_{t+1|t}\right) \\ P_{t+1|t+1} &= P_{t+1|t} - \gamma_{t+1}K_{t+1}CP_{t+1} \\ K_{t+1} &= P_{t+1|t}C'\left(CP_{t+1|t}C' + R\right)^{-1} \end{aligned}$$

Bounds for error convergence

Theorem 1. *If* $\left(A, Q^{\frac{1}{2}}\right)$ *is controllable,* (A, C) *is observable and* A *is unstable, there exists a* $\lambda_c \in (0, 1]$ *such that:*

$$\lim_{t\to\infty} \mathbb{E}\left[P_t\right] = +\infty, \qquad \textit{for } \; 0 \le \lambda \le \lambda_c \textit{ and } \exists P_0 \ge 0$$

$$\mathbb{E}\left[P_t\right] \le M_{P_0} \; \forall t \qquad \textit{for } \; \lambda_c < \lambda \le 1 \textit{ and } \forall P_0 \ge 0$$

where $M_{P_0} > 0$ *depends on the initial condition* $P_0 > 0$.

- The critical value λ_c cannot be directly calculated, however its upper and lower bound can
- When $\lambda > \lambda_c$, upper and lower bound for the expected value of state error covariance matrix exist

Bounds for error convergence

Critical value of arrival probability

- Lower bound

$$\underline{\lambda} = 1 - \frac{1}{\alpha^2}$$

where $\alpha = \max_i |\sigma_i|$ and σ_i are the eigenvalues of A.

- Upper bound –using bisection for the optimization problem

$$\overline{\lambda} = \text{argmin}\Phi_\lambda (Y, Z) > 0, \quad 0 \leq Y \leq I$$

where function $\Phi_\lambda (Y, Z)$ is defined:

$$\Phi(Y,Z) = \begin{bmatrix} Y & \sqrt{\lambda}(YA + ZC) & \sqrt{1-\lambda}YA \\ \sqrt{\lambda}\left(YA' + C'Z'\right) & Y & 0 \\ \sqrt{1-\lambda}A'Y & 0 & Y \end{bmatrix}$$

Bounds for error convergence

Bounds on the expected value of error covariance

- Lower bound

  $\overline{S} = (1-\lambda)\, A\overline{S}A'$

- Upper bound solving optimization problem

$$argmax_V \; Trace\,(V)$$

$$subject\,to$$

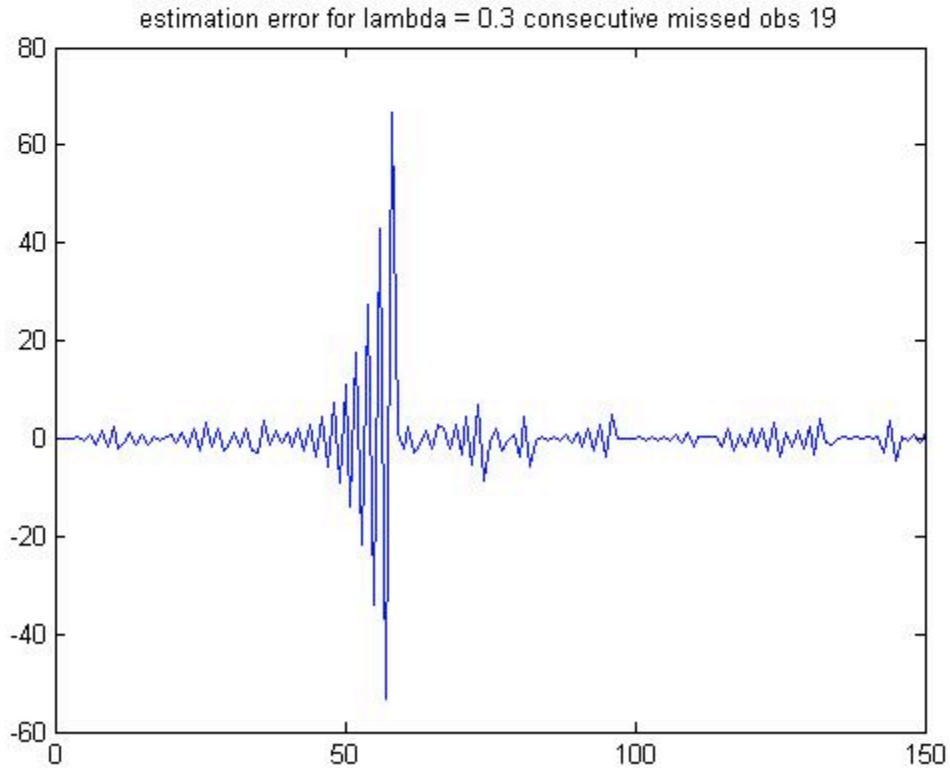
$$\begin{bmatrix} AVA' - V & \sqrt{\lambda}AVC' \\ \sqrt{\lambda}CVA' & CVC' + R \end{bmatrix} \geq 0,$$

$$V \geq 0.$$

Example

- Discrete unstable system with parameters
  $A = -1.25,\ C = 1$
  v_t and w_t zero mean and variance $R = 2.5$ and $Q = 1$

- Critical value of arrival probability 0.36



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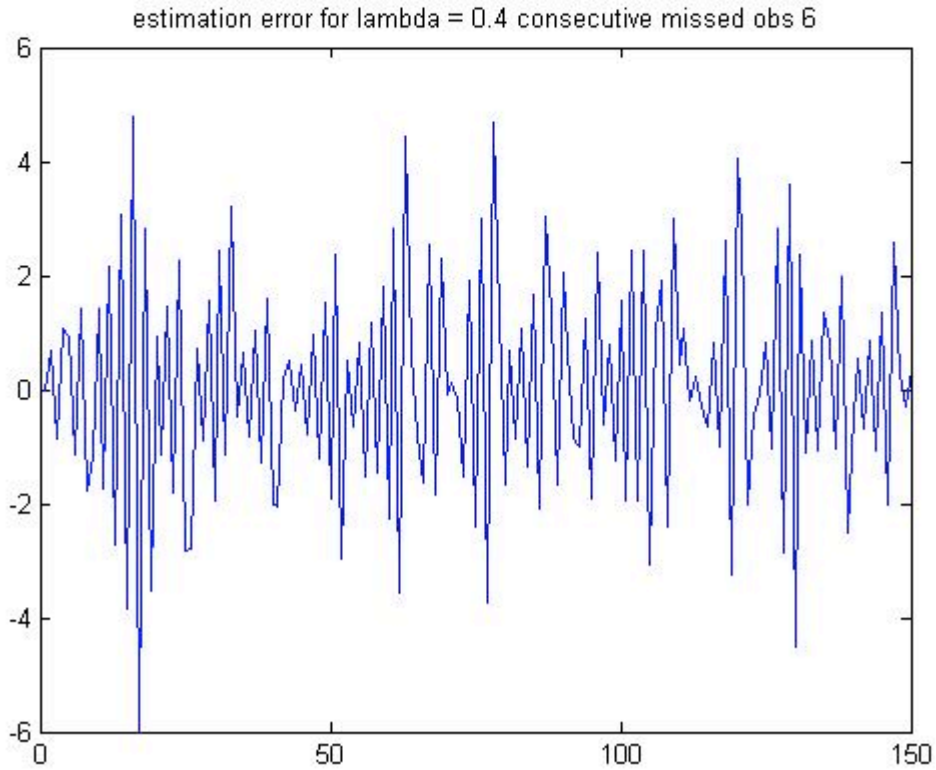

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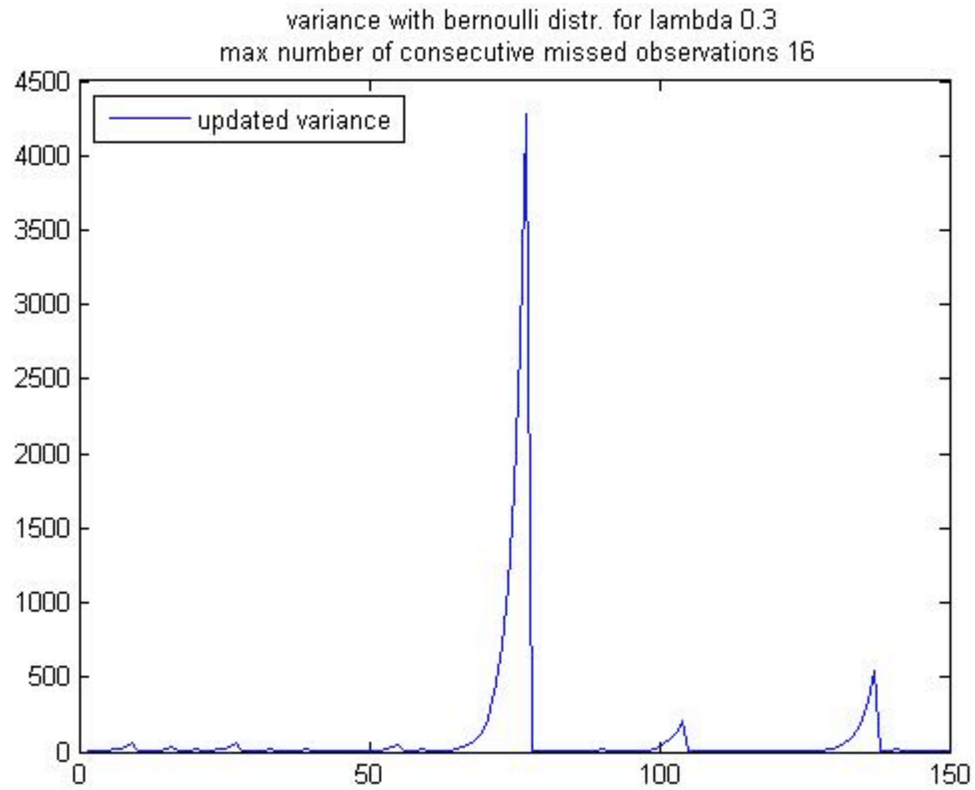

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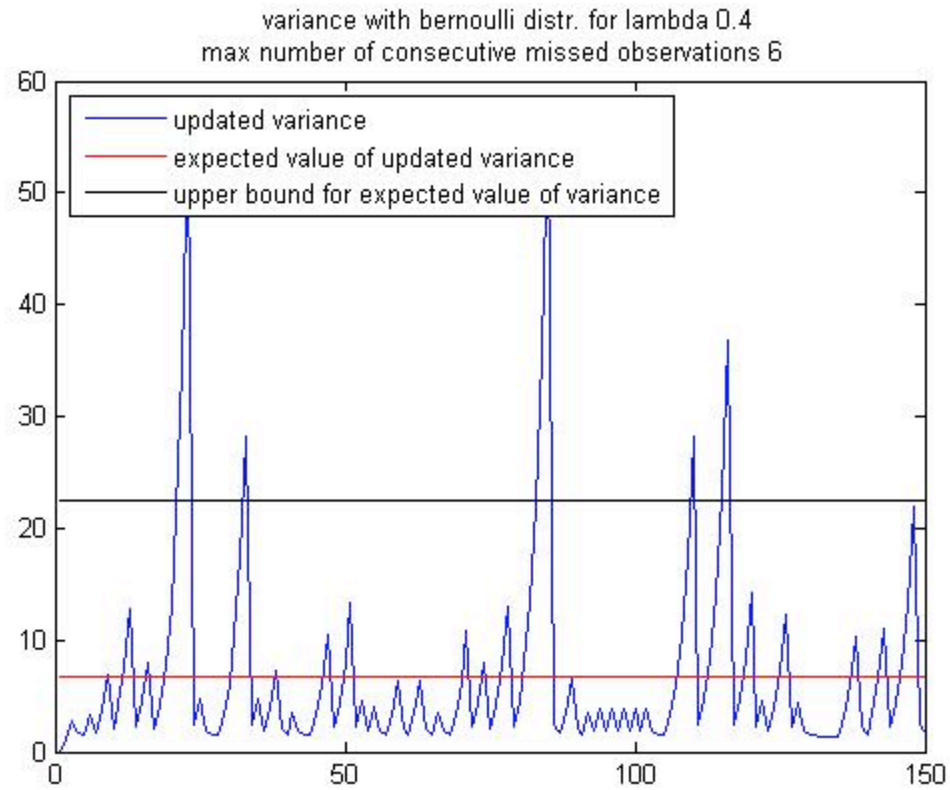


Heavy tailed distribution

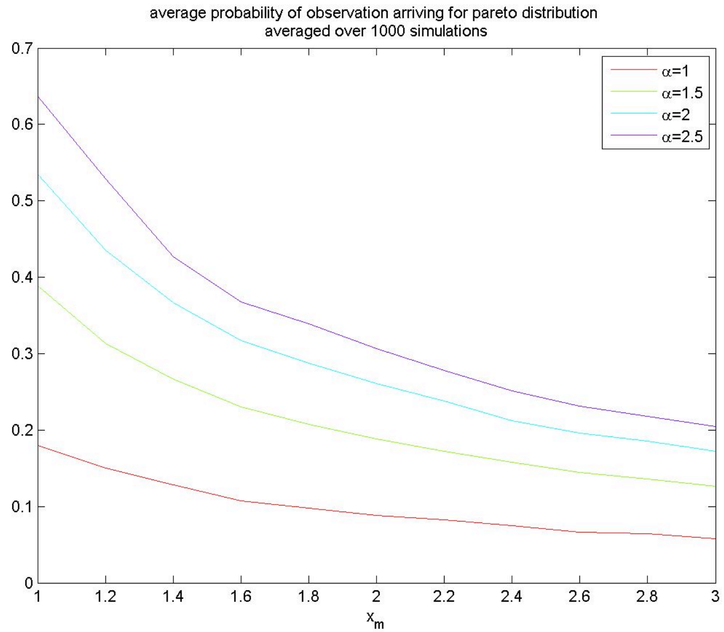
- With Bernoulli distribution burstiness of network traffic loss is not modeled appropriately
- Bit errors on telephone channels and losses in Ethernet and wireless networks all show heavy tail properties
- Heavy tailed distributions decay more slowly than exponential distribution
- Example: Pareto distribution:

  shape parameter α, location parameter x_m and pdf:

$$f_X(x) = \begin{cases} \alpha \frac{{x_m}^{\alpha}}{x^{\alpha+1}} & for\ x > x_m \\ 0 & for\ x < x_m \end{cases}$$

Heavy tailed distribution

- Model observation inter-arrival periods with Pareto
- Find average arrival probability λ_a
- Apply previous results and bounds using λ_a as λ
- Verify that the expected value of covariance matrix is bounded if $\lambda > \lambda_c$



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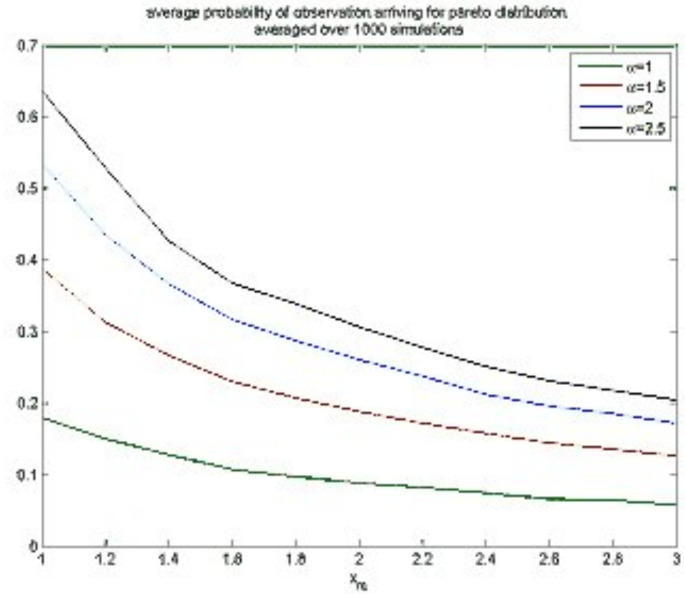

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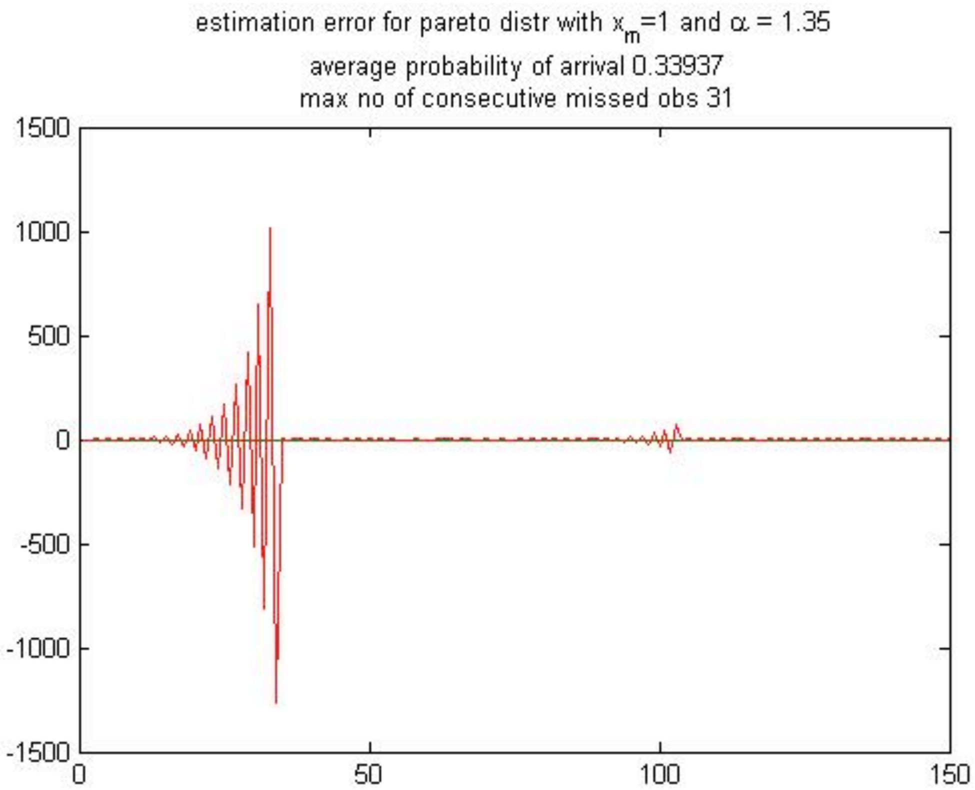

Example

- Discrete unstable system with parameters
  $A = -1.25$, $C = 1$
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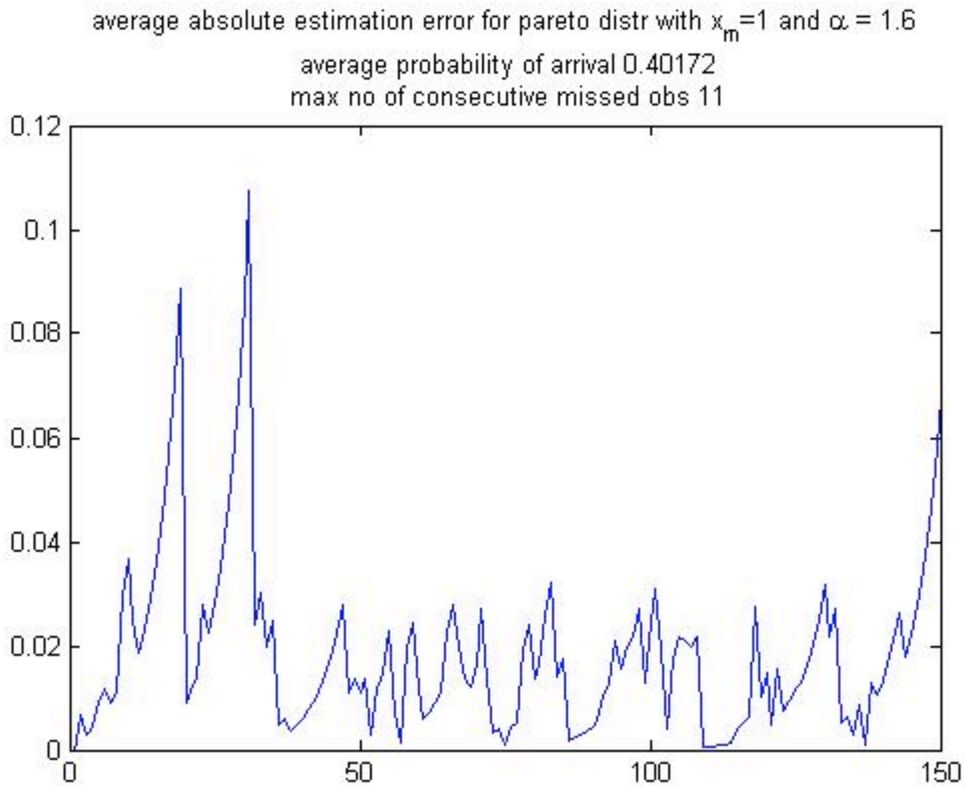

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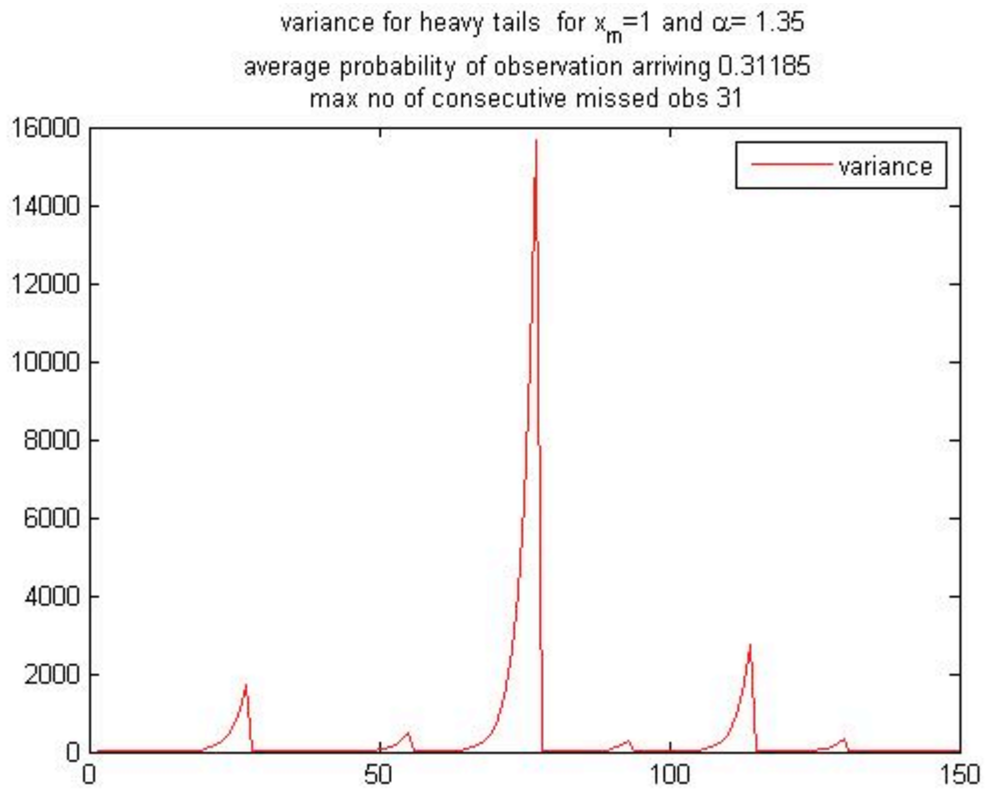

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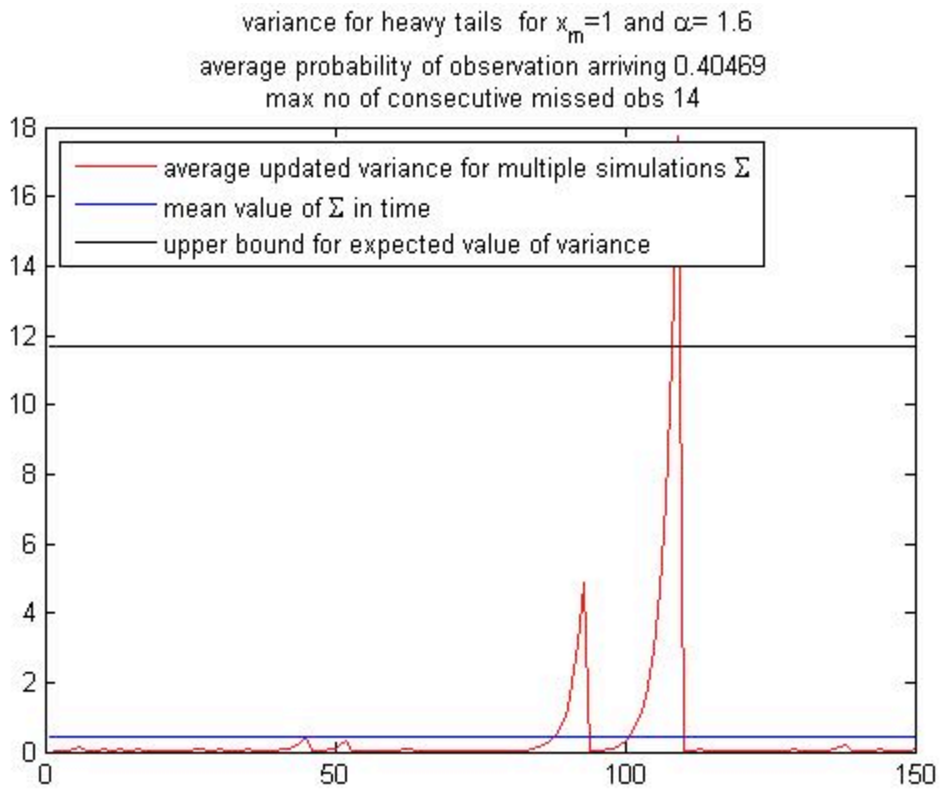

Conclusions

- Kalman filter with intermittent observations, arrival of observations can be modeled as Bernoulli random variables
- There exists a critical value of arrival probability of observations, above which state covariance remains bounded for all initial conditions
- Arrival of distributions can also be modeled with heavy tailed distribution
- If average arrival probability of Pareto modeled observations is above the critical value of arrival, error covariance stays finite