Kalman State Estimation over Lossy, asynchronous and randomly Delayed Sensor networks

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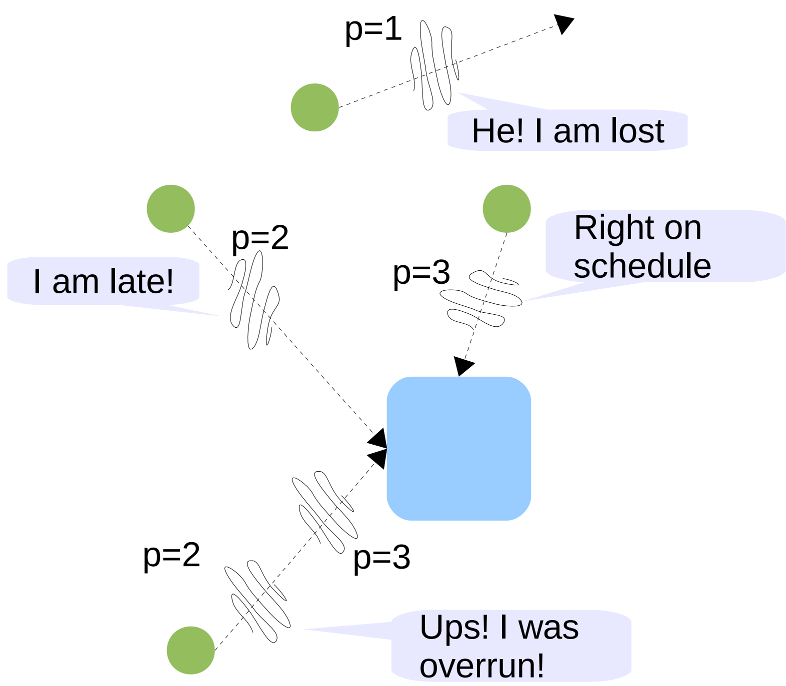
Minimum variance estimator

Problem Statement

Minimum variance Estimator

Main Properties

We are considering a network with asynchronous random delays and losses!



Sensor	t=0	t=1	t=2	t=3
1	p=0	-	p=1	p=2
2	-	p=0,p=1	-	p=3
3	-	-	p=1	p=0, p=2

To be able to solve the problem, we make all the 'usual' assumptions

- Assume that the plant ξ and measurement noise χ follow a zero mean Gaussian distribution ($N(0,R_{\xi,\xi})$,$N(0,R_{\chi,\chi})$)
- The initial state vector also follows follows a Gaussian distribution with mean E*a* and variance $R_{a,a}$
- The transmission delays are independent of the initial state and of the noise and have and upper bound, σ
- The transmission delays are bounded by a a maximum value \sigma – (all the ones with greater delays are considered lost)
- At each time t, the controller only has access to the control at t-1, and for the state estimation problem we assume that this value is always available

$x(t+1) = A(t)x(t) + B(t)u(t) + \xi(t),$ the state vector

$y_\nu(t) = C_\nu x(t) + \chi_\nu(t),$ the sensors output

$x(0) = a$ is the initial condition

$u(t)$ is the control - which we now at t+1

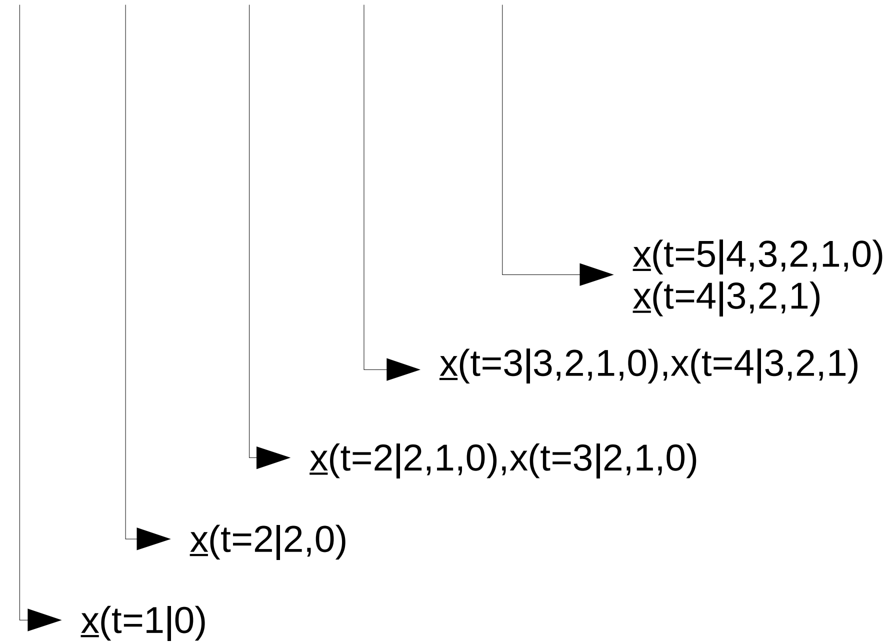
$Y(t) = (y_\nu(\theta))_{(\nu,\theta)\in S(t)}$ observations arriving at t

$S(t) = \{(\nu,\theta) : \theta + \tau_\nu = t\}$

And we want the minimum variance estimator for the state vector

T=	0	1	2	3	4	5
P=	0	-	2	1	3	4,5

Idea: use Kalman filter over all the observation in the interval t,t+σ



However, this would be very time consuming, and in this case we are interested in an online estimator, which could then be used for control

By augmenting the state vector to a whole interval [t,t+σ] we can re-use the KF

The augmented state vector dynamics is described by an extended state model

The augmented state vector and observations

$$X(t+1) = [x(t), x(t-1), ..., x(t-\sigma)]^T$$

$$Y(t) = (z_{\nu,j}) \in \zeta$$

$$z_{\nu,j} = \begin{cases} y_\nu(t-j) \text{if} & (\nu, t-j) \in S(t) \\ 0, & \text{otherwize} \end{cases}$$

The state model

$$X(t+1) = \mathcal{U}(t)X(t) + \mathcal{C}\xi(t)$$

$$X(0) = (a, ..., 0)^T$$

$$Y(t) = \mathcal{C}\,[t, S(t)]\,X(t) + \eta[t, S(t)]$$

$$\eta[t, S(t)] = (\mu_{\nu,\mathrm{j}})$$

$$\mathcal{U}(t) = \begin{pmatrix} A(t) & 0 & ... & 0 & 0 \\ I & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & I & 0 \end{pmatrix}$$

And we can plug this state model in a Kalman Filter to obtain a MVUE

Predict step:

$$X(t+1|t) = \mathcal{U}(t)X(t|t)$$

$$P(t+1|t) = \mathcal{U}P(t|t)\mathcal{U}^T + \eta[t, S(t)]$$

Update step:

$$X(t+1|t+1) = X(t|t) + \mathcal{K}(t+1)(Y[t, S(t)] - Y(t+1|t))$$

$$P(t+1|t+1) = P(t+1|t) - \mathcal{K}(t+1)\mathcal{C}P(t+1|t)$$

$$\mathcal{K}(t+1) = P(t+1|t)C^T\mathcal{S}(t+1)^+$$

$$\mathcal{S}(t+1) = \mathcal{C}P(t+1|t)\mathcal{C}^T + \mu$$

Since our problem is essentially sparse, we can rewrite and COMPUTE all of this is a much efficient way

The final expression looks more complex, but highlights the online nature of the estimator

The prediction stage is identical

$$\hat{x}(t+1|t) = A(t)\hat{x}(t|t) + B(t)u(t)$$
$$P_{0,0}(t+1|t) = A(t)P_{0,0}(t|t)A(t) + R_{\xi,\xi}(t),$$

But the update is cumbersome

$$\hat{x}(j|t+1) = x(j|t) + K^{(\nu,\theta)}_{t+1j}(t+1)[y_\nu(\theta)y\nu(\theta|t)]$$
$$j = t+1, t, ...t+1-\sigma$$
$$\hat{y}_\nu(\theta|t) := C_\nu(\theta)\hat{x}(\theta|t)$$
$$K^s_{t+1-j}(t) = \sum_{\nu,\theta\in S(t)} P_{j,t-\theta}(t)C_\nu(\theta)^T\lambda^+(t)^s_{\nu,\theta}$$
$$\lambda^+(t)^{s_2}_{s_1} = \mathcal{D}^{s_2}_{s_1}\lambda^+(t),$$
$$\text{except for all the empty rows and columns}$$
$$\lambda(t) = \begin{pmatrix} \lambda^{s_1}_{s_1}(t) & \lambda^{s_2}_{s_1}(t) & ... & \lambda^{s_q}_{s_1}(t) \\ \lambda^{s_1}_{s_2}(t) & \lambda^{s_2}_{s_2}(t) & ... & \lambda^{s_q}_{s_2}(t) \\ . & . & ... & . \\ . & . & ... & . \\ \lambda^{s_1}_{s_q}(t) & \lambda^{s_2}_{s_q}(t) & ... & \lambda^{s_q}_{s_q}(t) \end{pmatrix}$$
$$\lambda^{s_j}_{s_i}(t) = C_{\nu_1}(\theta_1)P_{t-\theta_1,t-\theta_2}C_{\nu_2}(\theta_2)^T + \nabla^{s_2}_{s_1},$$
$$\nabla^{s_2}_{s_1} = \begin{cases} R^{\nu_1}_{\xi,\xi}(\theta), & \text{if} \quad s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{P}_{i,j} = P_{i,j} - \sum_{(\nu,\theta)\in S(t)} K^{(\nu,\theta)}_i(t)C_\nu(\theta)P_{t-\theta,j}(\theta)$$
$$P_{i,j} = \begin{cases} \bar{P}_{0,0} & \text{, if} \quad i=0, j=0 \\ \bar{A}(t)P_{i,j-1}(t)^T & \text{, if} \quad i=0, j\geq 1 \\ \bar{P}_{i,j-1}(t)A(t)^T & \text{, if} \quad i\geq 1, j=0 \\ \bar{P}_{i,j-1} & \text{if} \quad i\geq 1, j\geq 1 \end{cases}$$

It is possible to use delayed measurements in the online situations required by control!

No need to order them,

This estimator is stable under 'natural' assumptions

'Natural' assumptions are similar to those we saw previously

Time invariant ORIGINAL system

$$A(t) \equiv A, \quad C_\nu(t) = C_\nu$$

All the process disturbances are statistically stationary and non singular

$$R^\nu_{\xi,\xi}(t) \equiv R^\nu_{\chi,\chi} > 0, \quad R_{\xi,\xi} \equiv R_{\xi,\xi} > 0$$

But we also require that the system is either stable or observable

A system with det A != 0 is observable via the communication system in the interval $[t_0,t_1]$ iff M(t-σ,t)>0

$$M(t_0,t_1) := \sum_{(\nu,\theta)\in S(t_0,t_1)}^{t_1} \sum_{\nu=1}^{l} \left(A^{\theta-t_0}\right)^T C_\nu^T C_\nu A^{\theta-t_0} > 0$$

In this case the system is stable in the Lyaponov sense

$$\mathcal{L}(t) = \mathcal{E}(t)\mathcal{P}^{-1}(t)\mathcal{E}(t) > 0$$

$$\Delta\mathcal{L}(t) \leq -\Big(V(t)^T P^{-1}(t)V(t)^T + \sum_{(\nu,\theta)\in S(t)} e(\theta|t)^T C_\nu^T e(\theta|t)\left(R^n_{\chi,\chi}\right)\Big) \quad \text{where:}$$

$$V(t) = \left(\bar{\mathcal{P}}(t)\mathcal{P}^{-1}(t) - I\right)\mathcal{U}\mathcal{E}(t-1)$$

$$\mathcal{E}(t+1) = \mathcal{U}\mathcal{E}(t) + V(t+1)$$

The problem in establishing stability, arises from the non-invariance of the observations equation of the system we are solving

Simulations

Problem Statement

Estimation results

Properties analysis

We simulated a car moving at constant velocity

Matrices for of the state model

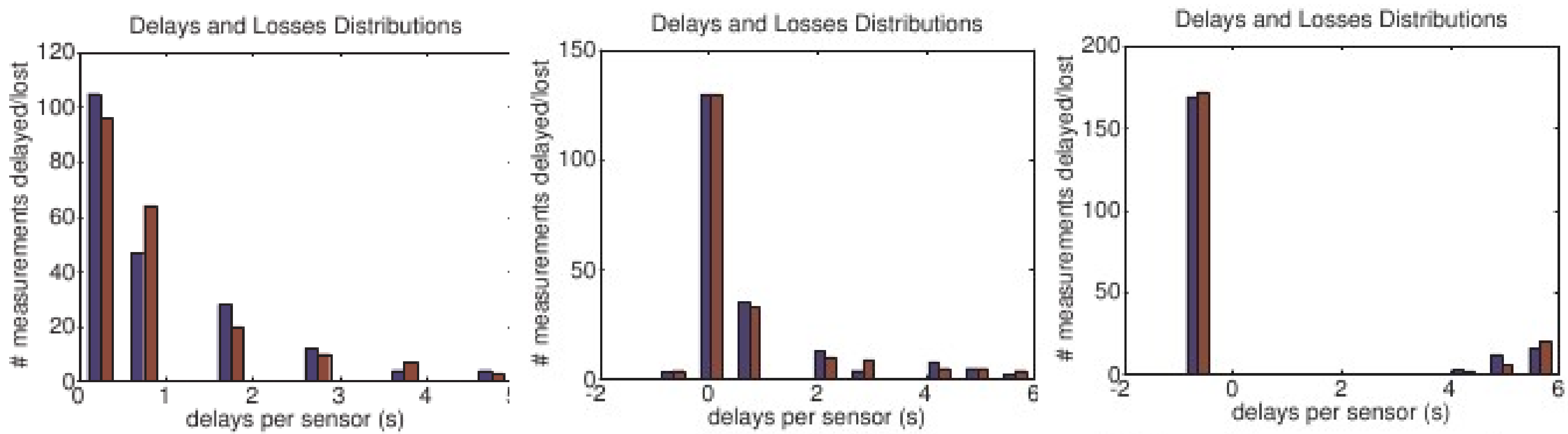
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R_{a,a} = 10^{-4} I_2$$

$$R_{\xi,\xi} = 10^{-4}$$

$$B = 0 \qquad R_{\chi,\chi} = 10^{-2}$$

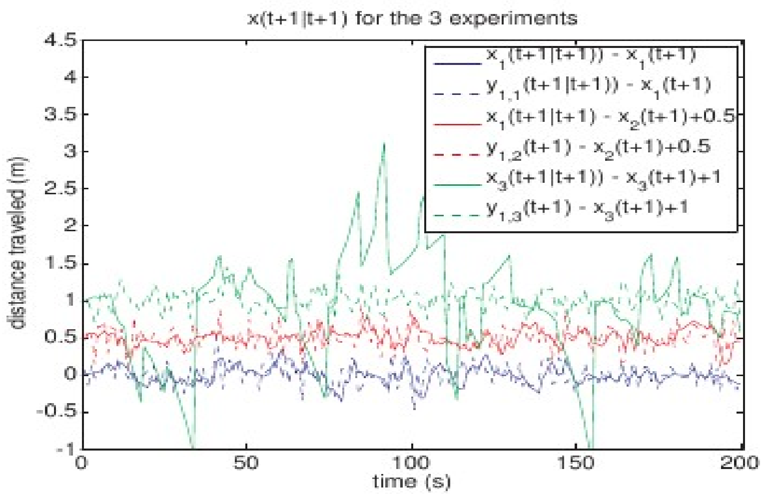
$$\mathbf{E}a = (0,0)^T$$

Delays histograms

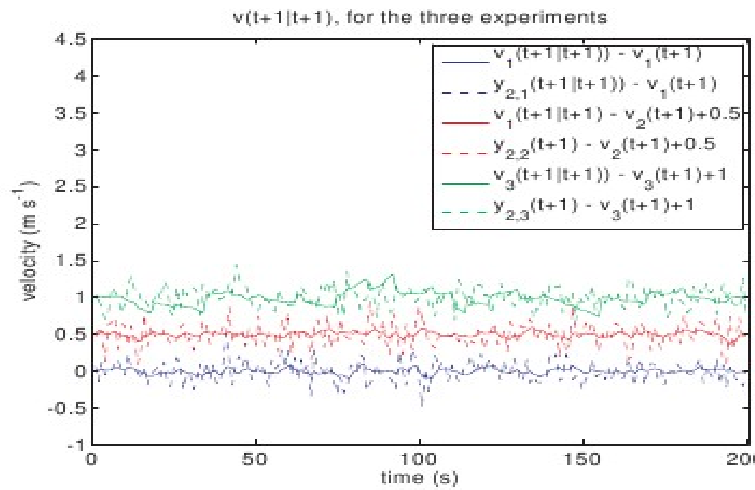


(a) All measurements are observable (b) Some measurements are lost (c) Most of the measurements are lost

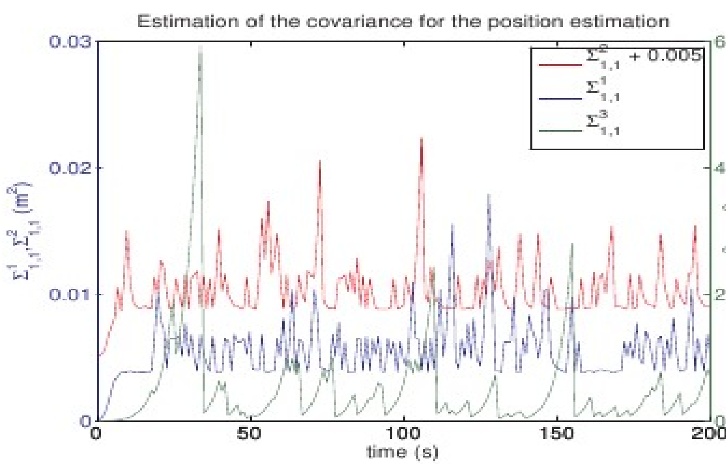
Results show the covariance matrix is bounded when we have no loss of data



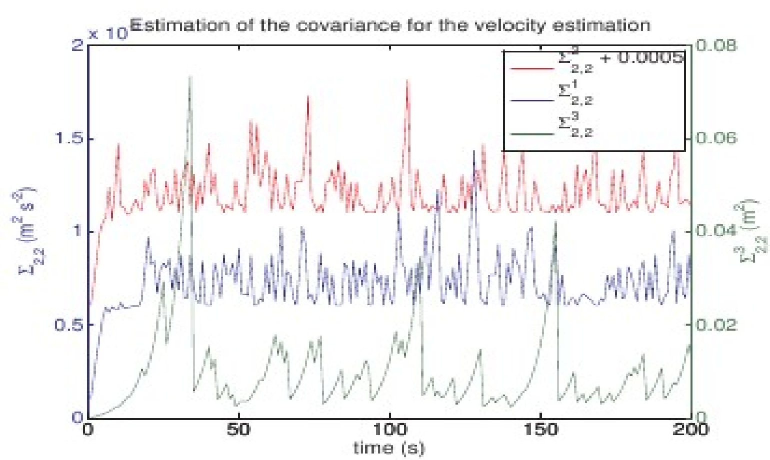
(a) Minimum variance estimation of the car position



(b) Minimum variance estimation of the car velocity



(c) Variance for the position

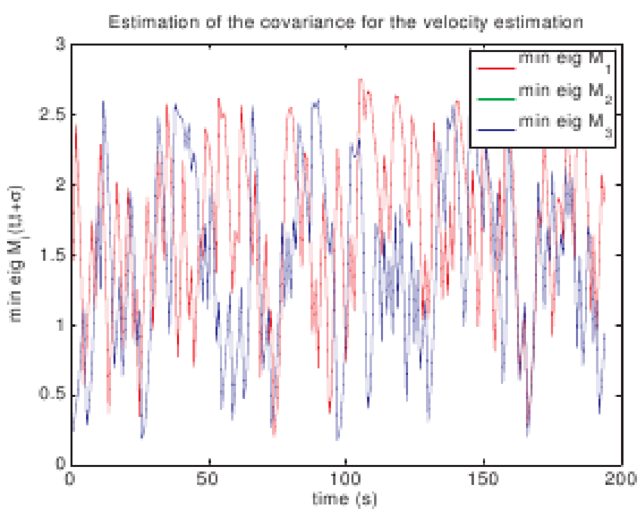


(d) Variance for the velocity

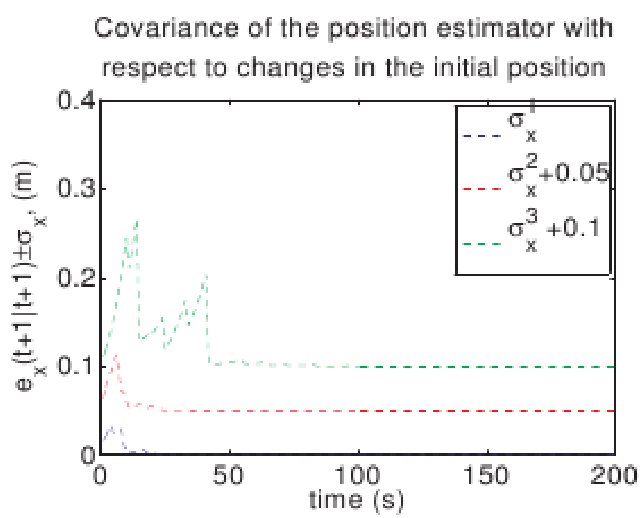
And also show that the system with many losses is not observable, but it is stable

Observability

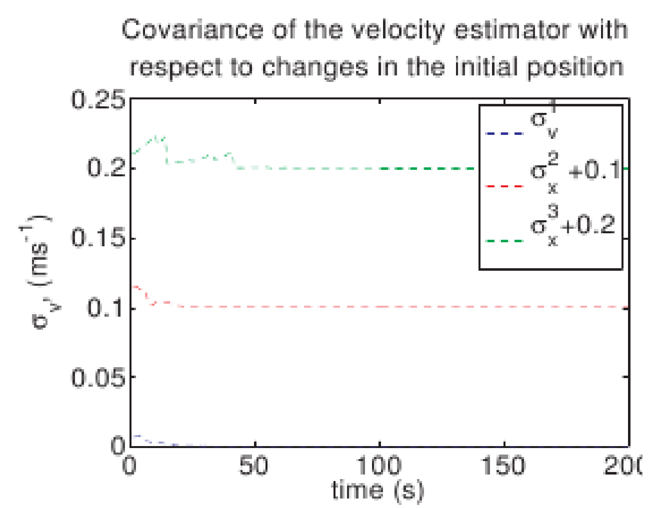
Stability



(a) Verification of the observability conditions: minimum eigen values of $M(t, t+\sigma)$



(b) Stability of the position estimator with respect to the initial position



(c) Stability of the velocity estimator with respect to the initial position

The third system is not observable via the communication channels

However, after some struggling it Is able to converge, since our system is stable!

Conclusions

The main advantage is the fastest way to integrate new observations

- **We can incorporate information of delayed observations online,**
  - The delayed observations still contribute to decrease the covariance!
  - Can be introduced without sorting
- **Is stable provided some basic assumptions**

Main disadvantage are the naïve considerations on the delays distribution

- **In real case systems, we rarely can assume the delays,**
  - One of the few examples is when the delay is caused by physical process which we may be able to estimate
- **If we lose the delays, this system is useless and a new approach should be provided**