State Vector estimation over asynchronous, randomly delayed and lossy measurements

Susana Brandão
Instituto Superior Técnico, Carnegie Mellon University
sbrandao@isr.ist.utl.pt

Abstract

The problem of state estimation in asynchronous, lossy and randomly delay systems arises frequently in problems connected with communications networks, where measurements may travel different paths between the plant which originated them and the controller. This usually leads to different delays between measurements of different sensors or even to the loss of some of the measurements. In particular, we consider the case where we have access to the control acting upon the plant and to the delays of each measurement arriving at time instant t, albeit we have no prior knowldge on the distribution of these delays. In this scenario we describe the minimum variance unbiased estimation of the state vector for an online situation described by Matveev and Savkin [3, 2]. We proceed by analysing the conditions for stability of the estimator and provide numererical simulations to better understand its constrains and the properties.

1. Introduction

Whenever communication between different sensors and a observer takes place over parallel and independent channels we are due to observe different delays between measurements. This is said to be an asynchronous system. In these systems, not only measurements corresponding to the same time t from different sensors may arrive at the observer at different times t' but also measurements from the same sensor may arrive out of order. Furthermore, as it happens often in communications over the internet, we may loose some of the data, due to noise or to protocol malfunctions. The diversity of causes which give rise to these delays and losses (see e.g. [4], [3]) makes their distribution difficult to model. Fortunately, it is common practice to provide the measurements with the corresponding time stamps and, if the observer and the producer have synchronous clocks or if it is possible to estimate the clocks independently (e.g. GPS/Gallileo satellite signals), the observer has direct access to the delays. These information allow us to place each new measurement with respect to previous ones and thus incorporate delayed measurements in our state vector estimations.

Usually, not only the measurements may be lost, but also the control. However, here we assume that we have access to the control, as is the case over TCP/IP networks. This assumptions is not only valid since it reflects a widely used protocol, but also it is useful in the sense that it allow us to separate the state estimation and the optimal control problem. Albeit here we only address the problem of state estimation, we keep in mind that the ultimate goal is to provide those estimations in an **online** fashion.

In this project, we present the minimum variance estimator for the state vector estimator of a system with asynchronous, lossy and delayed measurements, with a controller acting upon the system. The system follows all the *natural* assumptions of independent white Gaussian noise. This problem was already solved by Matveev and Savkin, [2]. Here we will provide an overview of the estimator itself as well as its main properties and will provide numerical examples to highlight those same properties. In particular, we will focus on the impact that delayed observations have on the estimation uncertainty and will adress the stability od the system.

This work is organized as follows: in section (2) we formalize the estimation problem and provide the MVUE; in section (3) we address the main characteristics of the estimator and finally, in section (4), we provide the numerical simulations.

2. State Vector Estimation

We start by describing our problem as a state model in its basic formulation: from plant state vector to observed measurements. Then we reformulate the problem as an augmented state model, to take into account the delays, and finally we present the state vector estimator based on this augmented model.

The state model of the sensor network can be described by the following equations for state vector dynamics, eq.(1), measurements, eq.(2) and initial conditions, eq.(3):

$$x(t+1) = A(t)x(t) + B(t)u(t) + \xi(t), \quad t = 0, ..., tmax \tag{1}$$

$$y_\nu(t) = C_\nu x(t) + \chi_\nu(t), \quad \nu = 1, ...l \tag{2}$$

$$x(0) = a, \tag{3}$$

where ν corresponds to sensor index. Furthermore, we are assuming independent white Gaussian noise for plant and measurements, which means that $\xi(t) \sim \mathcal{N}(0, R_{\xi,\xi})$ and $\chi(t) \sim \mathcal{N}(0, R_{\chi,\chi})$. Moreover, we assume that we have an uncertainty in the initial position and that this uncertainty is independent on both plant and measurements the noise and also follows a Gaussian distribution; $a \sim \mathcal{N}(\mathbf{E}a, R_{a,a})$. The noise arising from the communication between the sensor and the estimator can be incorporated under χ.

At each time instant t, the observer receives a set of measurements from different sensors ν and with different time stamps θ. To the set of all pairs (ν , θ) arriving in the instant t we call $S(t)$. Since we are considering the case where we have access to the producers (sensor) identity and the time stamp of each measurement, we can consider that measurements only arrive once $(S(t_1) \cap S(t_2) = \emptyset$ for $t_1 \neq t_2)$ and that have access to $S(t)$ at each time instant t. This means that our observations are independent and that we do not need the statistics of the delays and losses because we have first hand access to them.

The delay of each measurement is denoted as $\tau_n(t)$. These delays are produced independently of the plant and measurement noise. Moreover, we are assuming that the delays have an upper bound: $\tau_\nu(t) \leq \sigma, \quad \forall t \in [0, tmax] \forall \nu \in \{1, ..., l\}$. This upper bound is important because it allow us to restrain our problem: all the action will be happening in the time interval $[t - \sigma, t]$. To estimate the state vector, we will take advantage of this fact by introducing an augmented state model, where we will deal the whole system in this interval.

2.1. Augmented State Model

From the previous assumptions we know that for each time instant t all our new measurements will come from the interval $[t - \sigma, t]$. This lead the authors [2] to introduce a new state model which describes the system in this interval. The state vector and the observations in this model are given by equations (4) and (5)

$$X(t) = [x(t), x(t-1), ..., x(t-\sigma)]^T \tag{4}$$

$$Y(t) = (z_{\nu,j}) \in \zeta \tag{5}$$

$$z_{\nu,j} = \begin{cases} y_\nu(t-j) \text{if} & (\nu, t-j) \in S(t) \\ 0, & \text{otherwize} \end{cases} \tag{6}$$

and the state model is given by the equation set (7).

$$\begin{aligned}
X(t+1) &= \mathcal{U}(t)X(t) + \mathcal{B}u(t)\mathcal{H}\xi(t) \\
Y(t) &= \mathcal{C}\,[t, S(t)]\,X(t) + \eta[t, S(t)] \\
\eta[t, S(t)] &= \{\mu_{\nu,j}\}_{\nu=1,j=0}^{l\quad\sigma} \in \zeta \\
\mathcal{C}\,[t, S(t)] &= \{y_{\nu,j}\}_{\nu=1,j=0}^{l\quad\sigma} \in \zeta \\
y_{\nu,j} &= \begin{cases} C_\nu(t-j)x(j), & \text{if} \quad (\nu, t-j) \in S(t) \\ 0 & \text{otherwise} \end{cases} \\
\mathcal{U}(t) &= \begin{pmatrix} A(t) & 0 & ... & 0 & 0 \\ I & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & I & 0 \end{pmatrix} \\
\mathcal{B}(t) &= [B(t), 0, ...0]^T \\
X(0) &= (a, ..., 0)^T \\
\mathcal{P}(0) &= \begin{pmatrix} R_{a,a} & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & 0 & 0 \end{pmatrix},
\end{aligned} \tag{7}$$

where $\mathcal{P}(0)$ is the initial covariance matrix corresponding to the initial state vector $X(0)$. We can now introduce the Kalman filter to estimate $X(t+1)$ and this, according to Kalman Filter properties (see e.g. [1]), will be the minimum variance estimator for the whole set of state vectors $x(t+1), ...x(t+1-\sigma)$ conditioned over all the observations arriving until $t+1$ ($Y(0)...Y(t+1)$), the information regarding their origin ($S(0)...S(t+1)$) and the control information ($u(0)...u(t+1)$).

2.2. Kalman Filter for the Augmented State Model

By plugging in the previous state vector into a regular Kalman filter and taking advantage of the sparseness of matrix $\mathcal{U}(t)$ and observation vector $Y(t)$, the authors provide an efficient way to compute the state vector estimative.

The final expressions are much more spectacular than the actual concept which is behind them (the Kalman filter). Here we will briefly present them just to highlight the impact of each new delayed observation on the overall estimator.

The predictive phase of the filter corresponds to equations eq.(8) and eq.(9).

$$\hat{x}(t+1|t) = A(t)\hat{x}(t|t) + B(t)u(t) \tag{8}$$

$$P_{0,0}(t+1|t) = A(t)P_{0,0}(t|t)A(t) + R_{\xi,\xi}(t), \tag{9}$$

The update phase is a bit more cumbersome.

$$\hat{x}(j|t+1) = x(j|t) + K^{(\nu,\theta)}_{t+1j}(t+1)[y_\nu(\theta) y\nu(\theta|t)]$$
$$j = t+1, t, ...t+1-\sigma \qquad (10)$$

$$\hat{y}_\nu(\theta|t) := C_\nu(\theta)\hat{x}(\theta|t) \qquad (11)$$

$$K^s_{t+1-j}(t) = \sum_{\nu,\theta\in S(t)} P_{j,t-\theta}(t) C_\nu(\theta)^T \lambda^+(t)^s_{\nu,\theta} \qquad (12)$$

$$\lambda^+(t)^{s_2}_{s_1} = \mathcal{D}^{s_2}_{s_1}\lambda^+(t), \qquad (13)$$

except for all the empty rows and columns

$$\lambda(t) = \begin{pmatrix} \lambda^{s_1}_{s_1}(t) & \lambda^{s_2}_{s_1}(t) & ... & \lambda^{s_q}_{s_1}(t) \\ \lambda^{s_1}_{s_2}(t) & \lambda^{s_2}_{s_2}(t) & ... & \lambda^{s_q}_{s_2}(t) \\ . & . & ... & . \\ . & . & ... & . \\ \lambda^{s_1}_{s_q}(t) & \lambda^{s_2}_{s_q}(t) & ... & \lambda^{s_q}_{s_q}(t) \end{pmatrix} \qquad (14)$$

$$\lambda^{s_j}_{s_i}(t) = C_{\nu_1}(\theta_1) P_{t-\theta_1,t-\theta_2} C_{\nu_2}(\theta_2)^T + \nabla^{s_2}_{s_1}, \qquad (15)$$

$$\nabla^{s_2}_{s_1} = \begin{cases} R^{\nu_1}_{\xi,\xi}(\theta), & \text{if} \quad s_1 = s_2 \\ 0 & \text{otherwise} \end{cases} \qquad (16)$$

$$\bar{P}_{i,j} = P_{i,j} - \sum_{(\nu,\theta)\in S(t)} K^{(\nu,\theta)}_i(t) C_\nu(\theta) P_{t-\theta,j}(\theta) \qquad (17)$$

$$P_{i,j} = \begin{cases} \bar{P}_{0,0} & \text{, if} \quad i=0, j=0 \\ \bar{A}(t) P_{i,j-1}(t)^T & \text{, if} \quad i=0, j\geq 1 \\ \bar{P}_{i,j-1}(t) A(t)^T & \text{, if} \quad i \geq 1, j = 0 \\ \bar{P}_{i,j-1} & \text{if} \quad i\geq 1, j\geq 1 \end{cases} \qquad (18)$$

where s_i is a pair (ν_i, θ_i) and $\mathcal{D}^{s_j}_{s_i}$ a matrix such as $\lambda^{s_j}_{s_i} = D^{s_j}_{s_i}\lambda(t)$ except for all the empty rows and collumns.

As we can see, each new measurement $y_\nu(\theta)$ will contribute to update the estimative of all the state vectors in the interval $[t-\sigma, t]$, in particular it will correct the estimate $\hat{x}(t+1|t+1)$ even if we are not observing the measurements corresponding to $t+1$. This corresponds to update all the previous state vector estimatives $\hat{x}(j|\theta)$ (and thus reducing the covariance matrix) and then propagate the new estimatives and new covariances until $t+1$.

3. Stability of the Estimator

The main problem in establishing the stability of the estimator is that it was obtained from a system which is not time invariant: the augmented system. Even if we had a time invariant plant model, since the observation matrix $\mathcal{C}$ in the augmented model is not constant in time, we cannot establish stability through the steady state Kalman filter. However, there are some conditions which we can provide to make sure that the system is stable.

The extensive proof for the conditions of stability can be found in [2]. Here we will just provide the conditions and sketch the main steps of the proof. Further analysis on the effects of the main conditions on the stability will be provided in section 4.

The plant system must be time invariant: $C_\nu = C$ and $A(t) = A$

The noises are statistically stable and non Singular: $R_{\xi,\xi} > 0, R_{\chi,\chi} > 0$

Plant is either a.s. Observable via the communication channels or stable: According to the definition of observability over the communication channels ([2]) a system is observable in the interval $[t_0, t_1]$ if and only if it respects:

$$M(t_0, t_1) := \sum^{t_1}_{(\nu,\theta)\in S(t_0,t_1)} \sum^{l}_{\nu=1} \left(A^{\theta-t_0}\right)^T C^T_\nu C_\nu A^{\theta-t_0} > 0$$

where $S(t_0, t_1)$ is the set of measurements with time stamps $t_0 < \theta < t_1$.

If these conditions are met, the system is stable in the sense of Lyapunov, after some stabilizing time t^*. The full demonstration can be found in [3], but here we provide a brief overview of the main ideas behind it, in particular we introduce the Lyapunov function and provide evidence that it full fils the stability criteria. Before we proceed, we just introduce the estimation errors vector $\mathcal{E}(t)$, and the predicted and updated covariances of the augmented estimator $\mathcal{P}(t)$ and $\bar{\mathcal{P}}(t)$.

The Lyapunov function is defined as $\mathcal{L}(t) = \mathcal{E}(t)\mathcal{P}^{-1}(t)\mathcal{E}(t)$ and it is positive definite if and only if $\mathcal{P}^{-1}(t)$ is also. To show that this is true, we just need to show that $\mathcal{P}$ has a lower bound greater than zero and a finite lower bound. The lower bound is trivially the covariance matrix of a system with no delays and with no losses. The upper bound arrises from the observability via the communication channels constrain. Since the estimation error $x(t) - \hat{x}(t|t)$ is proportional to $M(t_0, t_1)^{-1}$, if the system is observable this matrix has an upper bound and so will $P(t) = \mathrm{E}[(x(t) - \hat{x}(t|t))^2]$. Furthermore the evolution along time of $\mathcal{L}$ can be computed by propagating $\mathcal{E}(t)$ in time and, by using the state model without noises and Kalman Filter Equations, we arrive at:

$$\Delta\mathcal{L}(t) \leq -\Big(V(t)^T P^{-1}(t) V(t)^T + \sum_{(\nu,\theta)\in S(t)} e(\theta|t)^T C^T_\nu \left(R^n_{\chi,\chi} C_\nu e(\theta|t)\right)\Big) \quad \text{where:}$$

$$V(t) = \left(\bar{\mathcal{P}}(t)\mathcal{P}^{-1}(t) - I\right)\mathcal{U}\mathcal{E}(t-1)$$
$$\mathcal{E}(t+1) = \mathcal{E}(t) + V(t+1)$$

and thus clearly $\Delta\mathcal{L}(t) \leq 0$ and the system is stable in the interval $[t_1 - \sigma, t_1]$.

4. Numerical simulations

The objective of this section is two fold: we wish to understand the impact of delays in both the estimation covariance and in stability. To achieve this, we consider only a linear time invariant system with three different delays and losses distributions.

The linear model emulates a car moving in a straight line moving under a constant acceleration. Furthermore the car is equipped with GPS, velocimeter and wifi, which allows him to communicate, through a wireless network, its position and velocity to an observer.

In the first distribution we consider no losses and small to none delays (see graphics in figure (1)). In the second distribution, we consider a small percentage of losses ($< 10\%$) and longer delays. At last we consider high losses and high delays.

4.1. Problem Description

The car state model can be described by the system in equations eq.(1), eq.(2) and eq.(3) with ;

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{19}$$

$$B = 0 \tag{20}$$

$$\mathbf{E}a = (0,0)^T \tag{21}$$

$$R_{a,a} = 10^{-4} I_2 \tag{22}$$

$$R_{\xi,\xi} = 10^{-4} \tag{23}$$

$$R_{\chi,\chi} = 10^{-2} \tag{24}$$

To simulate the delays and losses, we used a Weibull distribution for each of the sensors and considered that all measurements with delays greater than 6 were lost, i.e., $\sigma = 6$. To simulate each of the three types of delays, we used: $\tau_\nu^1(\theta) \sim 1.3\text{Wei}(1,1)$, $\tau_\nu^2(\theta) \sim 1.3\text{Wei}(0.7,1)$ and $\tau_\nu^3(\theta) \sim 1.3\text{Wei}(7,0.0001)$ respectively. These distributions were selected empirically in order to emulate the desired effect on the observations and have no physical meaning: we are considering the general case where we have no access to this distributions. The statistics for the delays are presented in the graphics of figure 1

4.2. State Vector estimation and uncertainty

The results from the simulations are presented in graphics of figure (2) for position, velocity, and variances respectively. The most interesting fact we can see in these experiments is the upper bound on the covariance of the systems which are observable over the communication system: the first and the second case. In the third experiment, our covariance achieves much higher values ($\sim$ 4 orders of magnitude higher).

We also note that the difference in magnitude between our uncertainty in velocity and in position is solely due to the structure of our state model.

4.3. Observability and Stability

If A is non singular, the condition for almost surely observability in an interval $[t_0, t_0 + \sigma]$ is equivalent to say that we can only lose data in a null measure (a la Lebesgue) set of points. In our examples this is only verified in the two first cases, when we have loss rates lower very low. To exemplify this, we computed the lower eigenvalue of $\mathcal{M}(t_1, t_1 + \sigma)$ for the three cases and the results are presented in the graphic of figure (ref). Only the third case, with high loss of measurements disrespect the observability over communication channels constrain ($M(t_1, t_1 + \sigma) > 0 \forall t_1 \in 0, ..., tmax - \sigma$).

According to the stability conditions, our system will only be stable on the first 2 cases, since in the third it does not fulfill the observability condition and the original plant system is not stable. This can be seen in the graphics of figure (3).

5. Conclusions

Here we presented the online minimum variance estimator for a system with random delays, in the case where we have access to them and when they are bounded. This emulates systems where both the observer and the plant have synchronized watches or at least where it is possible to estimate independently the delays and the state vector, as is the case in the TCP protocol. Furthermore, this estimator is able to overcome the stability issues intrinsically connected with the delays and neglectable losses of data.

However, this estimator lacks generality, because the two constrains which lead to its elegance of are too strong. In reality, computer clocks will not be synchronized and at most we can expect to be able to estimate the delays. The introduction of uncertainty would require a joint estimation of the delay and of the state vector, which is not possible under this framework. One sujestion for overcoming this drawback can be found at [4], where several models are considered: one for each type of problem in the comunications (e.g. one for modeling a delay of $\Delta t = 1$, another for $\Delta t = 2$ and other for modeling a measurement loss). Then given a new measurement the system should be able to predict not only the state model, but also the perturbation.

References

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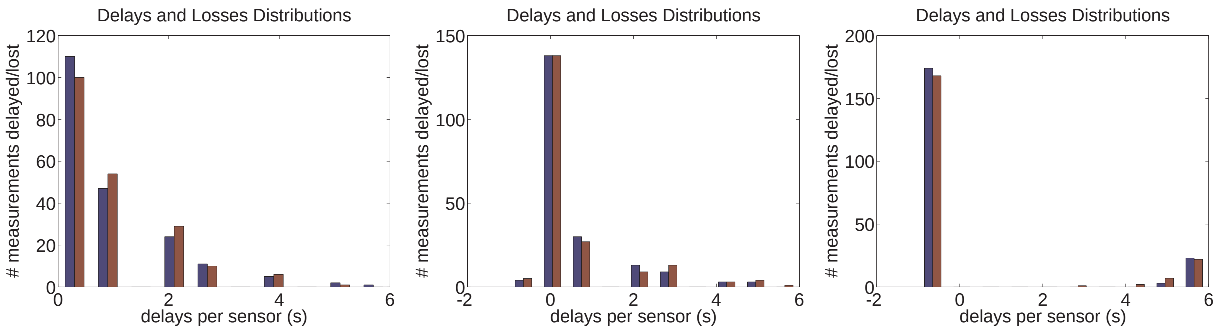
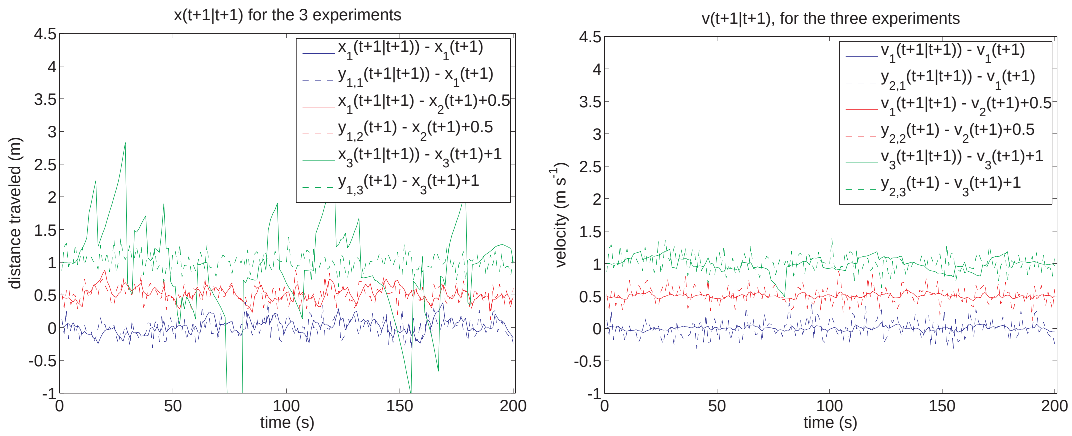
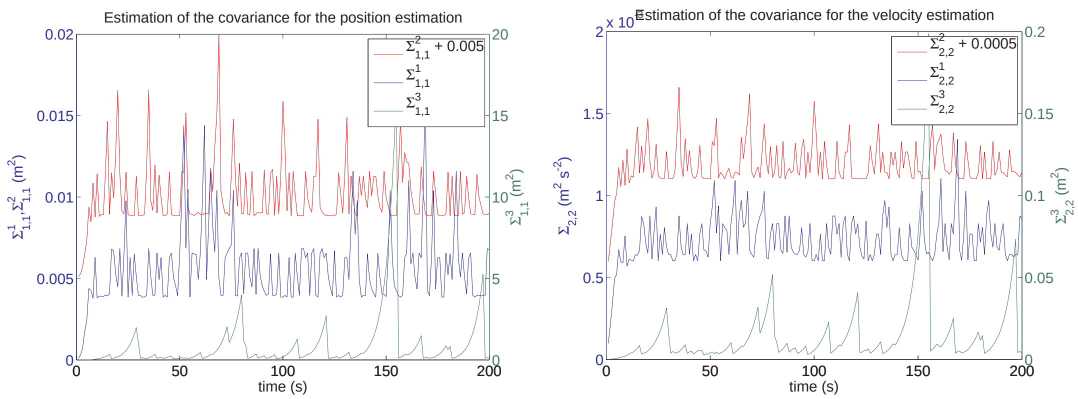


(a) All measurements are observable (b) Some measurements are lost (c) Most of the measurements are lost

Figure 1. Delays statistics for the 3 experiments. Delays labeled as -1 correspond to losses.



(a) Minimum variance estimation of the car position (b) Minimum variance estimation of the car velocity



(c) Variance for the position (d) Variance for the velocity

Figure 2. Results of estimation of the state vector and respective covariance matrix. A constant value was added to some of the variable to increase the graphics graphics. Whenever we had to do this, we indicated the respective constant value in the legend. Furthermore the covariances for the third experiment are related to the right side axis.

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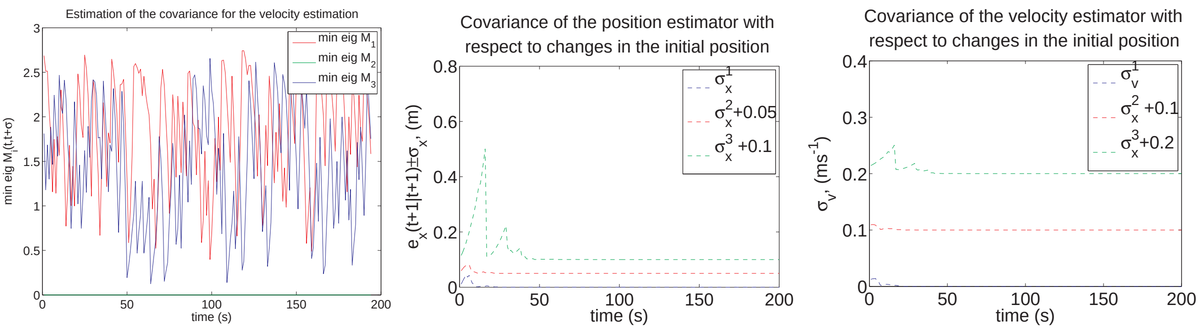

(a) Verification of the observability conditions: minimum eigen values of $M(t, t+\sigma)$

(b) Stability of the position estimator with respect to the initial position

(c) Stability of the velocity estimator with respect to the initial position

Figure 3. Estimator properties under different delay and loss statistics. Some of the plots were shifted in the vertical to allow for readibility. The value of the shift appears in the legend. In this plots we can see that the system with a great number of losses is not observable, but is stable with respect to initialization of the filter.