Observer design for a class of kinematic systems

Pedro Batista, Carlos Silvestre, Paulo Oliveira

Abstract—An observer design methodology is introduced for a class of kinematic systems that often arise in the development of Navigation Systems for vehicular applications. At the core of the proposed methodology there is a time varying orthogonal coordinate transformation that renders the observer error dynamics linear time invariant (LTI). The problem is then formulated as a virtual control problem which is solved by resorting to the standard $\mathcal{H}_\infty$ output feedback control synthesis technique, thus minimizing the $L_2$ induced norm from a generalized disturbance input to a performance variable. The resulting observer error dynamics are globally exponentially stable (GES) and several input-to-state stability (ISS) properties are derived. A relevant example is provided that demonstrates the potential and usefulness of the proposed design methodology and simulation results are offered to illustrate the filter achievable performance in the presence of extreme environmental disturbances and realistic sensors’ noise.

I. INTRODUCTION

This paper addresses the design of observers for a class of dynamic systems with direct application to the estimation of linear quantities in Integrated Navigation Systems. The observer design technique here proposed is motivated by previous work that can be found in [1], where resorting to a time varying orthogonal coordinate transformation the resulting observer error dynamics become linear time invariant (LTI).

Examples of application of the proposed observer can be foreseen in the design of accurate navigation and positioning systems for a great variety of mobile platforms. To tackle this class of problems several different approaches have been proposed in the literature. In [2] a passively globally exponentially stable (GES) observer for ships (in two-dimensions) that includes features such as wave filtering and bias estimation is presented and in [3] an extension to this result with adaptive wave filtering is available. An alternative filter was proposed in [4] where the problem of estimating the velocity and position of an autonomous vehicle in three-dimensions was solved by resorting to special bilinear time-varying complementary filters. A passivity based controller-observer design for $n$ degrees of freedom robots is proposed in [5] and a sliding mode observer for robotic manipulators is reported in [6]. The development of nonlinear observers for Euler-Lagrange systems has been addressed in [7] and [8]. In these problems it is often common to have available to the design only a subset of the desired physical quantities. Moreover, these measurements are usually corrupted by sensors’ noise and environmental disturbances such as the wind, sea waves, etc., that must be taken into account in the design phase. The particular problem of estimation of the linear quantities falls in the general framework presented in the paper, with direct application to the design of Integrated Navigation Systems.

The devised observer error dynamics are GES and several input-to-state stability (ISS) properties are derived that demonstrate the robustness of the solution relative to disturbances on the key physical quantities. The proposed design technique minimizes the $L_2$ induced norm from a generalized disturbance input to a performance variable, whereas the augmented observer error dynamics may include frequency weights to shape both the exogenous and the internal signals.

The present paper is organized as follows: Section II introduces the class of dynamic systems and the estimation problem addressed in this work. In Section III the proposed observer design technique is presented and some properties are derived in Section IV. A motivating example that demonstrates the potential and usefulness of the proposed design methodology is offered in Section V and simulation results are included to illustrate the filter achievable performance. Finally, Section VI summarizes the main contributions of the paper.

II. PROBLEM STATEMENT

Consider the class of dynamic systems

$$
\begin{align}
\dot{\eta}_1 &= f_1(t, \xi, \omega, \eta_1) + \gamma_1 \eta_2 \\
\dot{\eta}_2 &= f_2(t, \xi, \omega, \eta_1) + \gamma_2 \eta_3 - S(\omega) \eta_2 \\
& \quad \vdots \nonumber \\
\dot{\eta}_{N-1} &= f_{N-1}(t, \xi, \omega, \eta_1) + \gamma_{N-1} \eta_N - S(\omega) \eta_{N-1} \\
\dot{\eta}_N &= f_N(t, \xi, \omega, \eta_1) - S(\omega) \eta_N \\
\psi &= \eta_1
\end{align}
$$

(1)

where $\eta_i = \eta_i(t) \in X_i \subseteq \mathbb{R}^3, i = 1, \ldots, N$ are the system states, $\psi = \psi(t)$ is the system output, $f_i(\cdot), i = 1, \ldots, N$ are smooth functions of their arguments, $\xi$ and $\omega$ are parameterizing vectors, possible time varying, i.e., $\xi = \xi(t)$ and $\omega = \omega(t)$, $\gamma_i, i = 1, \ldots, N-1$, represents nonzero scalar constants, and $S(\omega)$ is a skew-symmetric matrix that verifies $S(a)b = a \times b$, with $\times$ denoting the cross product, and that satisfies $R = RS(\omega)$, where $R$ is a

This work was partially supported by Fundação para a Ciência e a Tecnologia (ISR/IST plurianual funding) through the projects PCT/MAR/55609/2004 - RUMOS of the FCT and MEDURES of the AdI. The work of P. Batista was supported by a PhD Student Scholarship from the POCTI Programme of FCT, SFRH/BD/24862/2005.

The authors are with the Institute for Systems and Robotics, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal. {pbatista, cjs, pjcro}@isr.ist.utl.pt
rotation matrix. The time dependence of $\eta$, $\psi$, $\xi$, $\omega$, and $R$ will be omitted in the sequel for the sake of simplicity.

The following assumption is made:

**Assumption 1:** The values of $R(t)$, $\xi(t)$, and $\omega(t)$ are available to be used in the observer. Moreover, $\omega(t)$ is bounded for all $t$, i.e., $\exists 0 < \omega < \infty$ for $\|\omega(t)\| < \omega$. The problem under discussion in the paper can be stated as follows.

**Problem Statement:** Consider the dynamic system (1) verifying Assumption 1. Design a state observer that minimizes the impact of sensor noise and external disturbances on the state estimates.

Throughout the paper the symbol 0 denotes a matrix of zeros, $I$ the identity matrix, both of appropriate dimensions, and diag($A_1, \ldots, A_N$) a block diagonal matrix.

### III. Observer Design

Consider an observer with the following structure

$$
\begin{align*}
\dot{\hat{\eta}}_1 &= f_1(t, \xi, \omega, \psi) + \gamma_1 \hat{\eta}_2 + S(\omega)(\psi - \eta_1) - \tau_1 \\
\dot{\hat{\eta}}_2 &= f_2(t, \xi, \omega, \psi) + \gamma_2 \hat{\eta}_3 - S(\omega) \hat{\eta}_2 - \tau_2 \\
\vdots & \quad \vdots \\
\dot{\hat{\eta}}_{N-1} &= f_{N-1}(t, \xi, \omega, \psi) + \gamma_{N-1} \hat{\eta}_N - S(\omega) \hat{\eta}_{N-1} - \tau_{N-1} \\
\dot{\hat{\eta}}_N &= f_N(t, \xi, \omega, \psi) - S(\omega) \hat{\eta}_N - \tau_N
\end{align*}
$$

where $\tau_i = \tau_i(t, \xi, \omega, \psi, \hat{\eta})$, $i = 1, 2, \ldots, N$, are virtual control variables that will be used to stabilize the observer error dynamics, with $\hat{\eta} = [\hat{\eta}_1^T, \hat{\eta}_2^T, \ldots, \hat{\eta}_N^T]^T$. Notice that, apart from the output injection term $S(\omega)(\psi - \eta_1)$, this structure is an exact copy of the nominal system. The reasoning behind the introduction of this term will become clear in the paper.

Let $\hat{\eta}_i = \eta_i - \hat{\eta}_i$, $i = 1, \ldots, N$, denote the state estimation errors. Hence, from (1) and (2), it follows that the observer error dynamics can be written as

$$
\begin{align*}
\dot{\hat{\eta}}_1 &= \gamma_1 \dot{\hat{\eta}}_2 - S(\omega) \dot{\eta}_1 + \tau_1 \\
\dot{\hat{\eta}}_{N-1} &= \gamma_{N-1} \dot{\hat{\eta}}_N - S(\omega) \dot{\eta}_{N-1} + \tau_{N-1} \\
\dot{\hat{\eta}}_N &= -S(\omega) \dot{\eta}_N + \tau_N
\end{align*}
$$

where $\dot{\eta} = [\eta_1^T, \eta_2^T, \ldots, \eta_N^T]^T$ and $S(\omega) = \text{diag}(S(\omega), \ldots, S(\omega))$. Notice that (3) is a Lyapunov transformation [9] as

- $T(t)$ is continuous differentiable for all $t$;
- Under Assumption 1 both $T(t)$ and $T(t)$ are bounded for all $t$, where $T(t) = T(t)M(\omega)$, with $M(\omega) := \text{diag}(S(\omega), \ldots, S(\omega));$
- $\det(T(t)) = 1$.

In the new coordinate space the observer error dynamics can be written as

$$
\dot{x}_p = A_p x_p + B_p T(t) \tau,
$$

where

$$
A_p = \begin{bmatrix}
0 & \gamma_1 I & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \gamma_{N-1} I \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix},
$$

$$
B_p = I, \quad \tau = [\tau_1^T, \tau_2^T, \ldots, \tau_N^T]^T.
$$

Applying the same coordinate transformation to the virtual control input of the observer error dynamics, i.e., $u = T(t) \tau$, allows to rewrite (4) as

$$
x_p = A_p x_p + B_p T(t) \tau, \quad \tau = [\tau_1^T, \tau_2^T, \ldots, \tau_N^T]^T.
$$

which is a linear time invariant system. Thus, with the coordinate transformation (3) the observer error dynamics are rendered LTI. The introduction of the term $S(\omega)(\psi - \eta_1)$ is now evident.

Naturally, not all the error states are available for feedback. In fact, only $x_1$ is accessible. Thus, to complete the observer error dynamics, define as output

$$
y_p := C_p x_p,
$$

where $C_p = [I \ldots I]$. Notice now that the LTI system (5)-(6) is both controllable and observable. Therefore, any design methodology for linear time invariant systems can be applied to stabilize the observer error dynamics, in particular the $H_\infty$ output feedback control synthesis. The employment of this design technique allows for the natural use of frequency weights to shape both the exogenous and the internal signals. To that purpose, consider Fig. 1, where the linear observer error dynamics are shown together with the disturbance weight matrix transfer functions $W_i(s)$, $i = 1, \ldots, 4$. In the figure, $w = [w_1^T w_2^T]^T$ and $z = [z_1^T z_2^T]^T$ represent the generalized disturbance and performance vectors, respectively. Notice that the models for the disturbance inputs and sensor noise are not exact as they live in the transformed space. The same applies to the performance weights.

**Fig. 1. Generalized Linear Observer Error Dynamics**

Define $x = [x_p^T x_{w1}^T x_{w2}^T x_{w3}^T x_{w4}^T]^T$, where $x_{w_i}, i = 1, \ldots, 4$, denote the states of the state space realizations of the frequency weights $W_i$, $i = 1, \ldots, 4$. Then, the augmented plant can be written, in a compact form, as

$$
\begin{align*}
\dot{x} &= A x + B_1 w + B_2 u \\
z &= C_1 x + D_{12} u \\
y &= C_2 x + D_{21} w,
\end{align*}
$$

where the definition of the various matrices is omitted as they are evident from the context. The standard design set-up and nomenclature in [10] is adopted and it is assumed that the $H_{\infty}$ control problem is well-posed. Let $T_{zw}(s)$ denote the closed-loop observer from the generalized disturbance vector $w$ to the generalized performance vector $z$. Then, the solution of the $H_{\infty}$ control problem for the augmented plant (7) yields a stabilizing compensator

$$\dot{x}_K = A_K x_K + B_K y,$$

that minimizes $\|T_{zw}(s)\|_{\infty}$. Combining (2) with (8) finally yields the observer in the original coordinates

$$\begin{align*}
\dot{\eta} &= f(t, \xi, \omega, \psi) + A_p \eta - M_s(\omega) \eta + C_p^T S(\omega) \psi - [T(t)]^T C_K x_K, \\
x_K &= A_K x_K + B_K R (\psi - \eta_1),
\end{align*}$$

where $f(t, \xi, \omega, \psi) = \begin{bmatrix} f_1(t, \xi, \omega, \psi)^T & \ldots & f_N(t, \xi, \omega, \psi)^T \end{bmatrix}^T$.

IV. OBSERVER PROPERTIES

In this section several properties of the proposed observer are presented and discussed. First, the asymptotic stability of the observer error is stressed in the following theorem.

**Theorem 1**: Consider the nominal dynamic system (1). Then, under Assumption 1, the error dynamics of the proposed nonlinear observer (9) are globally exponentially stable.

**Proof**: It has been established before that, with the proposed observer design, the observer error dynamics in the transformed coordinate space, $x_p$, are globally asymptotically stable. As the dynamics of $x_p$ are in fact linear time invariant, the convergence is exponentially fast. Now, using the fact that a Lyapunov coordinate transformation is employed, it follows that the original observer error, $\eta$, also converges exponentially fast to zero [9].

**Remark 1**: The previous result can also be established using the following Lyapunov function

$$V := \begin{bmatrix} \eta^T & x_K^T \end{bmatrix} P(t) \begin{bmatrix} \eta^T & x_K^T \end{bmatrix}^T,$$

with $P(t) := T_c^T(t) P_0 T_c(t)$, where

$$T_c(t) := \text{diag} \begin{bmatrix} T(t), \ldots, I \end{bmatrix},$$

and $P_0$ is the positive definite solution of the Lyapunov equation $A_c^T P_0 + P_0 A_c = -I$, where

$$A_c = \begin{bmatrix} A_p & B_p C_K \\ B_K C_p & A_K \end{bmatrix}.$$

The exponential behavior of the observer error dynamics is a very important property. Nevertheless, there exist GES systems that, in the presence of disturbances, even arbitrarily small vanishing disturbances, are driven to infinity [11]. The next results characterize the system with respect to errors in the various variables. First, an additional assumption is introduced.

**Assumption 2**: The function $f(t, \xi, \omega, \psi)$ is globally Lipschitz in $(\xi, \psi)$, i.e., $3_{0 \leq L_1 < \infty}$ such that

$$\|f(t, \xi_1, \omega, \psi_1) - f(t, \xi_2, \omega, \psi_2)\| < L_1 \left\| \begin{bmatrix} \xi_1 - \xi_2 \\ \psi_1 - \psi_2 \end{bmatrix} \right\|.$$

**Theorem 2**: Suppose that $\xi$ and $\psi$ in (9) are replaced by disturbed variables $\xi_m = \xi - \hat{\xi}$ and $\psi_m = \psi - \hat{\psi}$, where $\hat{\xi}$ and $\hat{\psi}$ denote the disturbances. Then, under the conditions of Theorem 1 and Assumption 2, the observer error is ISS from input $\begin{bmatrix} \xi^T & \psi^T \end{bmatrix}^T$.

**Proof**: Under the presence of disturbances in $\xi$ and $\psi$, the observer error dynamics can be written as

$$\begin{align*}
\dot{\eta} &= f(t, \xi, \omega, \psi) - f(t, \xi - \hat{\xi}, \omega, \psi) \\
&\quad + A_p \eta - M_s(\omega) \eta + C_p^T S(\omega) \psi + [T(t)]^T B_c K x_K,
\end{align*}$$

$$\begin{align*}
x_K &= A_K x_K + B_K R (\psi - \eta_1) - B_K R \hat{\psi}.
\end{align*}$$

Consider the Lyapunov-type function (10). Its derivative, under the presence of disturbances $\xi$ and $\psi$, satisfies

$$\dot{V} \leq - \left\| \begin{bmatrix} \hat{\eta} \\ x_K \end{bmatrix} \right\|^2 + L_V \left\| \begin{bmatrix} \hat{\eta} \\ x_K \end{bmatrix} \right\| \left\| \begin{bmatrix} \xi \\ \psi \end{bmatrix} \right\|,$$

with $L_V = 2 \|P_0\| [L_1 + W + \sigma_{\max}(B_c)]$, where $\sigma_{\max}(.)$ denotes the maximum singular value of a matrix. Let $0 < \theta < 1$. Then, it is easy to show that $\dot{V}$ verifies

$$\dot{V} \leq (1 - \theta) \left\| \begin{bmatrix} \hat{\eta} \\ x_K \end{bmatrix} \right\|^2 \forall \left\| \begin{bmatrix} \hat{\eta} \\ x_K \end{bmatrix} \right\| \geq \frac{L_V}{\theta} \left\| \begin{bmatrix} \xi \\ \psi \end{bmatrix} \right\|.$$

Since, in addition to (13), it can be shown that $V$ satisfies

$$\lambda_{\min}(P_0) \left\| \begin{bmatrix} \hat{\eta} \\ x_K \end{bmatrix} \right\|^2 \leq V \leq \lambda_{\max}(P_0) \left\| \begin{bmatrix} \hat{\eta} \\ x_K \end{bmatrix} \right\|^2,$$

it follows that the observer error is ISS from input $\begin{bmatrix} \xi^T & \psi^T \end{bmatrix}^T$ [12].

The additional presence of disturbances in $\omega$ and $R$ is addressed in the following theorem. Assumption 2 must be strengthened, as to include the Lipschitz condition in $\omega$ too. The new assumption is:

**Assumption 3**: The function $f(t, \xi, \omega, \psi)$ is globally Lipschitz in $(\xi, \psi, \omega)$, uniformly in $t$.

**Theorem 3**: Suppose that $\xi$, $\psi$, $\omega$ and $R$ in (9) are replaced by disturbed variables $\xi_m = \xi - \hat{\xi}$, $\psi_m = \psi - \hat{\psi}$, $\omega_m = \omega - \hat{\omega}$, and $R_m = R \left[ I - S(\hat{\lambda}) \right]$, where $\hat{\xi}$, $\hat{\psi}$, $\hat{\omega}$, and $\hat{\lambda}$ are the disturbances, respectively. Then, under the conditions of Theorem 1 and Assumption 3, the observer error is locally ISS, with $\begin{bmatrix} \xi^T & \psi^T & \hat{\omega}^T & \hat{\lambda}^T \end{bmatrix}^T$ as input.

**Proof**: The proof follows the same steps of Theorem 2 and is therefore omitted. The difference resides in the fact that only local ISS is now achieved.
Last, the optimality of the proposed observer design is addressed. To that purpose, consider the generalized observer error dynamics depicted in Fig. 2. The main difference between this generalized plant and the one of Fig. 1 is that: i) the generalized disturbances go through the transformation $T^T(t)$ and ii) the generalized performance vector takes into account the system states and the control signal after the transformation $T(t)$. In spite of these transformations, the magnitude of the signals is preserved - only the directional ity is affected over time. Let $\chi = [\eta^T x_{w,1}^T, \ldots, x_{w,2}^T]^T$. Notice that $\dot{\chi} = A(t)\chi + B_1(t)w + B_2(t)\tau$.

where

$$A(t) = -\begin{bmatrix} M_1(\omega) & 0 \\ 0 & 0 \end{bmatrix} + T_0^T(t)AT_c(t),$$

$B_1(t) = T_2^T(t)B_1$, and $B_2(t) = T_0^T(t)B_2T(t)$. The performance vector $\zeta$ can be written as

$$\zeta = C_1(t)\chi + D_{12}(t)\tau,$$

where $C_1(t) = C_1T_c(t)$ and $D_{12}(t) = D_{12}T(t)$. The generalized output is given by

$$\psi = C_2(t)\chi + D_{21}(t)w,$$

where $C_2(t) = R^T(t)C_2T_c(t)$ and $D_{21}(t) = R^T(t)D_{21}$. The following theorem addresses the optimality of the proposed solution.

**Theorem 4:** Under the conditions of Theorem 1, the proposed observer minimizes the $L_2$ induced norm from $w$ to $\zeta$, assuming that $w$ is a finite energy signal, i.e., $w$ is square integrable.

**Proof:** Suppose that $w \in L_2$, where $L_2$ denotes the set of real-valued finite energy signals, and consider the closed-loop systems from $w$ to $\zeta$ and from $w$ to $\zeta$. Let $\gamma^*$, associated to the control input $\tau^*(t)$, be the minimum $\gamma$ that satisfies

$$\int \zeta^T(t)\zeta(t)dt \leq \gamma^2 \int w^T(t)w(t)dt,$$

and $\gamma^*_l$, associated with the control input $u^*(t)$, be the minimum $\gamma_l$ that satisfies

$$\int z^T(t)z(t)dt \leq \gamma^*_l \int w^T(t)w(t)dt.$$

Notice that $u^*(t)$ is the control law resulting from the $H_\infty$ synthesis. Choosing $\tau(t) = T^T(t)u^*(t)$, it is easy to show that

$$\int \zeta^T(t)\zeta(t)dt \leq \gamma^*_l \int w^T(t)w(t)dt \leq \gamma^*_l \int \gamma^*_l \int w^T(t)w(t)dt,$$

where

$\zeta = T_c(t)w$.

from which one concludes that $\gamma^* \leq \gamma^*_l$. On the other hand, choosing $u(t) = T(t)\tau^*(t)$, it is easy to show that

$$\int z^T(t)z(t)dt \leq \gamma^* \int w^T(t)w(t)dt \leq \gamma^* \int \gamma^* \int w^T(t)w(t)dt,$$

from which one concludes that $\gamma^*_l \leq \gamma^*$. Since $\gamma^* \leq \gamma^*_l$ and $\gamma^*_l \leq \gamma^*$ it must be $\gamma^* = \gamma^*_l$ with $\tau^*(t) = T^T(t)u^*(t)$. Thus, the proposed observer minimizes the $L_2$ induced norm from $w$ to $\zeta$.

It is important to remark that the observer structure was previously imposed and did not arise naturally from the solution of an optimization problem. Nevertheless, good performance can be achieved with the minimization of the $L_2$ induced norm from $w$ to $\zeta$ in the augmented error dynamics depicted in Fig. 2, as it will be clearly demonstrated in the next section.

**V. SIMULATION RESULTS**

This section presents a case study of practical interest in marine applications that demonstrates the potential and usefulness of the proposed observer design methodology. This problem was first posed in [1].

Consider an Underwater Vehicle equipped with an acoustic positioning system like an Ultra Short Base Line (USBL) and suppose that there is a moored buoy in the mission scenario where an acoustic transponder is installed. The linear velocity kinematics of the vehicle can be written as $\dot{p} = Rv$, where $p$ is the position of the origin of the body-fixed coordinate system $\{B\}$ described in the inertial coordinate system $\{I\}$, $R$ is the rotation matrix from $\{B\}$ to $\{I\}$, that verifies $R = RS(\omega)$, $v$ is the linear velocity of the vehicle relative to $\{I\}$, expressed in body-fixed coordinates, and $\omega$ is the angular velocity, also expressed in body-fixed coordinates. Assume that the buoy where the transponder is installed is subject to wave action of known power spectral density that affects its position over time, and suppose that the position of the vehicle with respect to the transponder is available, in body-fixed coordinates as measured by an USBL sensor installed onboard. Suppose also that the body angular velocity $\omega$ and the rotation matrix $R$ are available from an Attitude and Heading Reference System (AHRS). Finally, suppose that the vehicle is moving in deep waters (far from the wave action), in the presence of an ocean current of constant velocity, which expressed in body-fixed coordinates is represented by $\nu_c$. The problem considered here is that of estimate the velocity of the current and the position of the vehicle with respect to the transponder. Further consider that the velocity of the vehicle relative to the water is available from the measures of an onboard Doppler velocity log. In shallow waters, this sensor can be employed to measure both the velocity of the vehicle relative to the inertial frame and relative to the water. However, when the vehicle is far from the bottom the inertial velocity is usually not available.
By estimating the ocean current velocity, an estimate of the velocity of the vehicle relative to the inertial frame is immediately obtained.

Let \( \mathbf{e} \) denote the position of the transponder and \( \mathbf{v}_r \) denote the velocity of the vehicle relative to the fluid, both expressed in body-fixed coordinates. Since the position of the transponder is assumed constant (in the absence of environmental disturbances) in the inertial frame, the time derivative of \( \mathbf{e} \) is given by \( \dot{\mathbf{e}} = -\mathbf{v}_r - \mathbf{v}_r - \mathbf{S}(\omega)\mathbf{e} \). On the other hand, as the velocity of the fluid is assumed to be constant in the inertial frame, the time derivative of this quantity expressed in body-fixed coordinates is simply given by \( \dot{\mathbf{v}}_r = -\mathbf{S}(\omega)\mathbf{v}_c \). Notice that the velocity of the fluid relative to the inertial frame satisfies \( \mathbf{v} = \mathbf{v}_r + \mathbf{v}_c \). Clearly, the problem of estimating the velocity of the fluid, \( \mathbf{v}_c \), falls into the class of problems addressed in the paper. To make it explicit, just consider the system (1) with \( \eta_1 = \mathbf{e} \), \( \eta_2 = \mathbf{v}_c \), \( \xi = \mathbf{v}_r \), \( \xi_1(t, \xi, \omega, \eta_1) = -\xi - \mathbf{S}(\omega)\eta_1 \), \( \gamma_1 = -1 \), \( \xi_2(t, \xi, \omega) = 0 \), and \( N = 2 \). Thus, it is possible to design an observer as detailed in Section III. Note that, in this case, the position of the transponder changes with time as the latter is assumed to be mounted in a buoy moored close to the sea surface, subject to strong wave action. The buoy wave induced random motion can be modeled as errors in the USBL positioning system expressed in the inertial frame, and their description embedded in the frequency weights presented in Section III. As closed loop design objective consider the rejection of the wave induced disturbances from the position measurements to the position and current velocity estimates. The disturbances induced by the three-dimensional wave random fields in the position of the buoy are modeled using three second-order harmonic oscillators representing the disturbance models along the \( x \), \( y \) and \( z \) directions, \([2], [13]\)

\[
H^i(s) = \frac{\sigma_i s}{s^2 + 2\xi_i \omega_0 i s + \omega_0^2 i^2}, \; i = 1, 2, 3,
\]

where \( \omega_0 i \) is the dominating wave frequency along each axis, \( \xi_i \) is the relative damping ratio, and \( \sigma_i \) is a parameter related to the wave intensity. In the simulation the dominating wave frequency was set to \( \omega_0 i = 0.8975 \text{rad/s} \) and the relative damping ratio to \( \xi_i = 0.1 \). Thus, the sensor frequency weight matrix transfer function \( \mathbf{W}_2(s) \) was chosen as

\[
\mathbf{W}_2(s) = 5 \left( 1 + \frac{\sigma_1 s}{s^2 + 2\xi_1 \omega_0 i s + \omega_0^2 i^2} \right) \mathbf{I}_3.
\]

Notice that a direct term was added, not only to satisfy the requirements of the \( \mathcal{H}_\infty \) design but also to model the errors of the position sensor, which were assumed Gaussian with standard deviation of 1 m.

As the observer nominal model, that corresponds to the kinematics of the linear motion, is exact and there is neither model uncertainty nor state disturbances the weight \( \mathbf{W}_1(s) \) was set to \( \mathbf{W}_1(s) = 0.01 \mathbf{I}_6 \). Using the fact that this is a pure disturbance rejection control problem, the performance weights were selected as \( \mathbf{W}_3(s) = \mathbf{I}_6 \). Finally, the virtual control input weights were chosen as \( \mathbf{W}_4(s) = 2(s+1)/(s+10) \mathbf{I}_6 \) to properly tune the input-output behavior of the closed loop system.

Fig. 3 shows the singular values of the linear closed loop transfer functions from the position error measurements in the inertial frame, signal \( \mathbf{n} \) in Fig. 1, to the position and current velocity estimate errors in the inertial frame, \( \mathbf{R}\mathbf{e} \) and \( \mathbf{R}\mathbf{v}_c \), respectively. The diagram shows that the performance requirements are met by the resultant closed loop system, which is evident from the band rejection characteristics of the notch filter present in both singular value diagrams. The structure of the resulting observer is depicted in Fig. 4, where the \( \mathcal{H}_\infty \) output feedback compensator is of order 18.

To illustrate the performance of the proposed observer a simulation was carried out with a simplified model of the SIRENE underwater vehicle \([14]\). In addition to the disturbances induced by ocean waves, which were confined to intervals of about 10 m of amplitude, and the disturbances of the USBL positioning sensor, in the simulation the measurements of the velocity of the vehicle relative to the water and the angular velocity were also assumed to be corrupted by Gaussian noise, with standard deviations of 0.01 m/s and 0.02 rad/s, respectively.

The time evolution of the observer estimates is presented in Fig. 5. The position of the buoy if there were no ocean waves is also shown, as well as the actual velocity of the fluid, all expressed in body-fixed coordinates. From these plots the performance of the observer is already evident - only the initial transients are noticeable. The evolution of the observer error is shown in Fig. 6. As the initial

![Fig. 3. Singular values of the closed loop system](image1)

![Fig. 4. Position and current velocity observer structure](image2)

![Fig. 5. Actual (dash-dot lines) and estimated (solid lines) variables](image3)
transients are of no interest - they arise due to the mismatch of the initial conditions of the states of the observer and can be considered as a warming up time of 180s of the corresponding Integrated Navigation System - the observer error is shown in more detail in Fig. 7. From the various plots it can be seen that the disturbances induced by the waves, as well as the sensors’ noise, are highly attenuated by the observer, producing very accurate estimates of the velocity of the current and the position of the buoy. To conclude this case study, it should be noticed that, if the position of the transponder in the inertial frame, at rest, is known to the vehicle, then, an estimate of the actual position of the vehicle in the inertial frame is simply given by $\mathbf{p} = \mathbf{i}(\mathbf{e}) - \mathbf{R}\mathbf{e}$, where $\mathbf{i}(\mathbf{e})$ is the position of the transponder expressed in the inertial frame. Fig. 8(a) depicts the actual and the estimated vehicle trajectory. For comparison purposes, the non-filtered position of the vehicle is plotted on Fig. 8(b). It is clear how accurate the observer estimates the trajectory described by the vehicle, even in the presence of severe wave action affecting the position of the buoy and realistic sensors’ noise.

**VI. Conclusions**

This paper presented an observer design methodology for a class of kinematic systems with application to the design of Integrated Navigation Systems. At the core of the proposed design technique there is a time varying orthogonal coordinate transformation that renders the observer error dynamics linear time invariant (LTI). The problem was then cast into a virtual control problem that was solved resorting to the standard $\mathcal{H}_\infty$ output feedback controller design technique. The resulting observer error dynamics were shown to be globally exponentially stable (GES) and several input-to-state stability (ISS) properties were derived. The proposed design technique minimizes the $\mathcal{L}_2$ induced norm from a generalized disturbance input to a performance variable, whereas the augmented observer error dynamics may include frequency weights to shape both the exogenous and the internal signals.

A case study of practical interest in marine applications was presented that demonstrates the potential and usefulness of the proposed observer design methodology. Simulation results were offered that illustrate the filter achievable performance in the presence of extreme environmental disturbances and realistic sensors’ noise.

**REFERENCES**


