A Globally Exponentially Stable Solution for Frequency Estimation

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Abstract—This paper proposes a globally exponentially stable (GES) discrete time observer for estimating the constant unknown parameters describing a single biased continuous time sinusoidal signal. A discrete-time dynamic system is derived based on the sampled output that will be key to the design of the estimation solution. The observability properties of this system are analyzed and a filter design is proposed which guarantees that the estimation error converges globally exponentially fast to zero. Realistic simulation results are presented, in the presence of measurement noise, that illustrate the performance of the proposed solutions. The performance of the achieved solution is also assessed in the estimation of the parameters of a linear chirp signal.

I. INTRODUCTION

The problem of estimating the constant unknown parameters describing a single biased sinusoidal signal received the attention of the scientific community in several areas, namely in the rejection of oscillations in control systems design, in non-coherent demodulation in telecommunications, in underwater acoustics, and in many other signal processing problems.

A variety of approaches to several variants of these classes of problems can be distinguished mainly by the estimation tools and accuracy, computational complexity, and processing latency. A popular approach is based on the Fast Fourier Transform (FFT) [1], however relying on blocks of data with increasing length for higher accuracy. The latency associated with performing these computations after the signal been received precludes real time scenarios.

An approach also very widespread in the community consists on the use of maximum likelihood estimators (MLE) to obtain one or several of the parameters describing the sinusoidal signal under consideration, see [2] and references therein. The estimators based on the Maximum Likelihood have a Gaussian distribution asymptotically. No guarantees on efficiency or on the required length of number of samples is known for many of those problems, thus Monte Carlo techniques are required to be used to characterize the estimator obtained.

Linear regression based techniques are accurate at moderate signal to noise ratios, with the additional benefit of low computational complexity. Moreover, given the reduced number of parameters to be estimated, low latency occurs due to the fact that the processing is performed as the samples arrive in real time [3], [4]. However, as only some of the problems can be formulated linearly on the unknowns, linearization or expansion approaches have been proposed, with no performance or convergence guarantees [5]. On the same vein as above, the use of extended Kalman filters for the nonlinear estimation problems at hand [6], [7]. However, it is well known that those type of tools and approaches fail on stability/convergence and performance guarantees, in special on low signal to noise scenarios.

Adaptive filters where all signals are globally bounded and that estimate the frequency asymptotically can be found in [8]. A simple tuning procedure is proposed, which trades-off the adaptation tracking capabilities with noise sensitivity. However, the transient performance can only be assessed resorting to extensive simulations. Another adaptive observer is proposed in [9], where Input-to-State Stability is used to compute bounds on the higher order harmonics and on the unstructured disturbance influence on the frequency estimates.

This work departs from the previous approaches as it provides an online estimator for unknown constant amplitude, frequency, phase, and bias on a single sinusoidal signal. The approach proposed tackles he problem at hand resorting to a dynamic model with extra states featuring globally exponentially stability (GES) of the discrete time observer obtained. The constant unknown parameters describing the sampled biased single sinusoidal signal. The analysis of the observability properties of th new system are instrumental for the filter design, such that the estimation error converges globally exponentially fast to zero.

This paper is organized as follows: Section II introduces the model and the problem to be solved. Section III presents the derivation and analysis of an estimation solution for the frequency of the sinusoidal signal (1). In section IV simulation results are presented and discussed to illustrate the proposed method for frequency. Finally, in section V some conclusions are drawn.
A. Notation

The symbols $\mathbf{0}$ represents a matrix (or vector) of zeros and $\mathbf{I}_n$ denote the $n \times n$ identity matrix. When $n$ is omitted, the matrices are of appropriate dimensions, which can be inferred from the context. A block diagonal matrix is represented by $\text{diag}(A_1, \ldots, A_n)$.

II. Problem Statement

Consider a sinusoidal signal of the form

$$s(t) = A \sin(\omega t + \phi) + B,$$

where $A > 0$ is the amplitude of the oscillations, $B \in \mathbb{R}$ is an offset, $\omega = 2\pi f$, where $f > 0$ is the frequency of the sinusoidal signal, and $\phi$ denotes its phase. Suppose that this signal is periodically sampled with sampling frequency $f_s = 1/T_s$, as given by

$$y(k) = s(t_k) = A \sin(\omega t_k + \phi) + B, \quad (1)$$

where $t_{k+1} = t_k + T_s$ and $T_s > 0$ is the sampling period. The problem considered in this paper is, given (1), to design an estimation solution for $\omega$ whose error converges to zero, exponentially fast, for all initial conditions.

III. Estimation Solution

This section presents the derivation and analysis of an estimation solution for the frequency of the sinusoidal signal (1). First, a discrete-time dynamic system is derived in Section III-A based on the available output that will be key to the design of the estimation solution. The observability properties of this system are analyzed in Section III-B, and the filter design follows in Section III-C. Finally, given the estimates provided by the filter proposed in Section III-C for the system derived in Section III-A, frequency estimates are provided in Section III-D.

A. System design

Define a first system state as

$$x_1(k) := s(t_k),$$

which is measured. The difference equation that describes the evolution of $x_1(k)$ can be written, using (1) and

$$\sin \theta - \sin \psi = 2 \sin \left(\frac{\theta - \psi}{2}\right) \cos \left(\frac{\theta + \psi}{2}\right), \quad (2)$$

as

$$x_1(k+1) = x_1(k+1) - x_1(k) + x_1(k)$$

$$= x_1(k) + A [\sin(\omega t_{k+1} + \phi) - \sin(\omega t_k + \phi)]$$

$$= x_1(k) + 2A \sin \left(\frac{\omega T_s}{2}\right) \cos \left(\omega t_k + \phi + \frac{\omega T_s}{2}\right).$$

Now, define a second system state as

$$x_2(k) := 2A \sin \left(\frac{\omega T_s}{2}\right) \cos \left(\omega t_k + \phi + \frac{\omega T_s}{2}\right),$$

such that the evolution of $x_1(k)$ is simply given by

$$x_1(k+1) = x_1(k) + x_2(k).$$

Next, compute the dynamics of $x_2(k)$, as given by

$$x_2(k+1) = x_2(k+1) + x_2(k) - x_2(k)$$

$$= x_2(k)$$

$$+ 2A \sin \left(\frac{\omega T_s}{2}\right) \cos \left(\omega t_{k+1} + \phi + \frac{\omega T_s}{2}\right)$$

$$- 2A \sin \left(\frac{\omega T_s}{2}\right) \cos \left(\omega t_k + \phi + \frac{\omega T_s}{2}\right)$$

$$= x_2(k) - 4A \sin^2 \left(\frac{\omega T_s}{2}\right) \sin(\omega t_{k+1} + \phi). \quad (3)$$

where the trigonometric identity

$$\cos \theta - \cos \psi = -2 \sin \left(\frac{\theta + \psi}{2}\right) \sin \left(\frac{\theta - \psi}{2}\right)$$

was used. Using (1) in (3) allows to write

$$x_2(k+1) = x_2(k) - 4A \sin^2 \left(\frac{\omega T_s}{2}\right) [s(t_{k+1}) - B].$$

Now, define two new system states as

$$x_3(k) := -4A \sin^2 \left(\frac{\omega T_s}{2}\right), \quad (4)$$

and

$$x_4(k) := 4B \sin^2 \left(\frac{\omega T_s}{2}\right), \quad (5)$$

such that the evolution of $x_2(k)$ is simply given by

$$x_2(k+1) = x_2(k) + s(t_{k+1}) x_3(k) + x_4(k).$$

Next, notice that both $x_3(k)$ and $x_4(k)$ are constant, hence

$$\begin{cases} x_3(k+1) = x_3(k) \\ x_4(k+1) = x_4(k) \end{cases}.$$

Let

$$x(k) := \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}.$$

Then, it is possible to write the system dynamics

$$\begin{cases} x(k+1) = A(k)x(k) \\ y(k+1) = Cx(k+1) \end{cases}$$

with

$$A(k) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & s(t_{k+1}) & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$
B. Observability analysis

Before designing a state estimator for (6) it is convenient to first characterize its observability properties. For that, notice that the system matrix $A(k)$ depends on the signal $s(t_{k+1})$, which is measured. This is not a problem for the characterization of the observability of (6), and observer design, since $s(t_{k+1})$ is available, and hence $A(k)$ can be seen as any other known matrix that is a function of the time. This was formally established, for the discrete-time case, in [10] and hence it suffices to analyze the observability of the pair $(A(k), C)$. Moreover, this approach has been successfully pursued in the past in the field of navigation of autonomous vehicles, see e.g. [10] and [11]. A sufficient observability for (6) is established in the following theorem.

**Theorem 1:** Suppose that $s(t_{k+1}) \neq s(t_{k+2})$. Then, the system (6) is observable on $[k_0, k_0 + 4]$, in the sense that the initial state $x(k_0)$ is uniquely determined by the output $\{y(k) : k = k_0, k_0 + 1, k_0 + 2, k_0 + 3\}$.

**Proof:** The proof consists in showing that the observability matrix $O(k_0, k_0 + 3)$ associated with the pair $(A(k), C)$ on $[k_0, k_0 + 4]$ has rank equal to the number of states of the system, i.e. rank 4, which immediately gives the desired result by application of [10, Lemma 1]. Suppose that the rank of the observability matrix is less than 4. Then, there exists a unit vector

$$d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \in \mathbb{R}^4,$$

with $d_1, d_2, d_3, d_4 \in \mathbb{R}$, such that $O(k_0, k_0 + 3) \cdot d = 0$ or, equivalently,

$$\begin{cases} \text{Cd} = 0 \\
CA(k_0)d = 0 \\
CA(k_0 + 1)A(k_0)d = 0 \\
CA(k_0 + 2)A(k_0 + 1)A(k_0)d = 0 
\end{cases}.$$  \hfill (7)

Expanding the first equation of (7) implies that it must be $d_1 = 0$. Substituting that in the second equation of (7) implies that it must be $d_2 = 0$. Expanding the third and fourth equation of (7) and substituting $d_1 = d_2 = 0$ gives

$$E\begin{bmatrix} d_3 \\ d_4 \end{bmatrix} = 0,$$  \hfill (8)

with

$$E = \begin{bmatrix} s(t_{k_0+1}) & 1 \\ 2s(t_{k_0+1}) + s(t_{k_0+2}) & 3 \end{bmatrix} \in \mathbb{R}^{2 \times 2}.$$  

Now, notice that the determinant of $E$ is given by

$$\det(E) = s(t_{k_0+1}) - s(t_{k_0+2}).$$

Hence, in order for (8) to hold, it must be

$$s(t_{k_0+1}) = s(t_{k_0+2}).$$

By contraposition, if $s(t_{k_0+1}) \neq s(t_{k_0+2})$, the rank of the observability matrix $O(k_0, k_0 + 3)$ is 4 and hence the system is observable, thus concluding the proof.

The previous result yields a technical sufficient condition for observability of (6). However, the output of this system is not completely arbitrary. In fact, the output of this system, for the purposes set in this paper, is always of the form of (1). Thus, it is of interest to derive conditions that depend on the signal parameters such that the system is observable. The next lemma addresses this issue and yields a result for observability that has a clear interpretation in light of well-known sampling results.

**Lemma 1:** Consider the system (6) with output identical to (1). If the sampling frequency $f_s$ is such that

$$f_s > 2f,$$  \hfill (9)

then there exists an infinite number of discrete time instants $k_0 \in \mathbb{Z}$ such that the system (6) is observable on $[k_0, k_0 + 4]$, in the sense that the initial state $x(k_0)$ is uniquely determined by the output $\{y(k) : k = k_0, k_0 + 1, k_0 + 2, k_0 + 3\}$.

**Proof:** The proof amounts to show that there exists an infinite number of discrete-time instants $k_0$ such that, if (9) holds, then $s(t_{k_0+1}) \neq s(t_{k_0+2})$. The conclusion then follows from Theorem 1. Assume that (9) holds. That, together with the fact that both $\omega$ and $T_s$ are positive, allows to conclude that

$$0 < \omega T_s \leq \pi.$$  \hfill (10)

Now, consider an arbitrary $k \in \mathbb{Z}$, fix $k_0 = k$, and write the signal for $k_0 + 1$ and $k_0 + 2$, as given by

$$s(t_{k_0+1}) = A \sin(\omega t_{k+1} + \phi) + B$$  \hfill (11)

and

$$s(t_{k_0+2}) = A \sin(\omega t_{k+2} + \phi) + B = A \sin(\omega t_{k+1} + \phi + \omega T_s) + B.$$  \hfill (12)

For the sake of simplicity, choose $n \in \mathbb{Z}$ such that $\theta \in [0, 2\pi[$, with

$$\theta := \omega t_{k+1} + \phi - 2n\pi.$$  \hfill (13)

Considering the trigonometric identity

$$\sin(x) = \sin(x + 2n\pi), \ n \in \mathbb{Z}, x \in \mathbb{R},$$

and using (13) in (11) and (12) gives

$$s(t_{k_0+1}) = A \sin(\theta) + B$$  \hfill (14)

and

$$s(t_{k_0+2}) = A \sin(\theta + \omega T_s) + B,$$  \hfill (15)

respectively. If $s(t_{k_0+1}) \neq s(t_{k_0+2})$, the proof follows from Theorem 1. Suppose now that $s(t_{k_0+1}) = s(t_{k_0+2})$, which means that $k_0 = k$ is not a suitable choice. That implies, from (14) and (15), that $\sin(\theta) = \sin(\theta + \omega T_s)$ or, equivalently,

$$\sin(\theta + \omega T_s) - \sin(\theta) = 0$$

$$2 \sin(\frac{\omega T_s}{2}) \cos(\theta + \frac{\omega T_s}{2}) = 0,$$

where (2) was used. Given (10), it follows that it must be

$$\cos(\theta + \frac{\omega T_s}{2}) = 0.$$  \hfill (16)
In this case, consider another candidate $k_0 = k + 1$ and write the output for $k_0 + 1$ and $k_0 + 2$, as given by
\[
s(t_{k_0+1}) = A \sin(\omega t_{k+2} + \phi) + B = A \sin(\omega t_{k+1} + \phi + \omega T_s) + B
\]
and
\[
s(t_{k_0+2}) = A \sin(\omega t_{k+3} + \phi) + B = A \sin(\omega t_{k+1} + \phi + 2\omega T_s) + B,
\]
respectively. Using (13) and $\sin(x) = \sin(x + 2\pi n)$ these can be rewritten as
\[
s(t_{k_0+1}) = A \sin(\theta + \omega T_s) + B
\]
and
\[
s(t_{k_0+2}) = A \sin(\theta + 2\omega T_s) + B. \tag{18}
\]
Now, take the difference between (18) and (17), as given by
\[
s(t_{k_0+2}) - s(t_{k_0+1}) = A \left[ \sin(\theta + 2\omega T_s) - \sin(\theta + \omega T_s) \right]
\]
which completes the proof. Indeed, for any $k \in \mathbb{Z}$, either $k_0 = k$ or $k_0 = k + 1$ satisfy the conditions of Theorem 1.

Albeit the proof of Lemma 1 is rather technical (and boring), the idea is very simple and intuitive: if a sinusoidal signal is periodically sampled with a sampling frequency higher than twice the frequency of the sinusoidal signal, then there do not exist three consecutive sampling instants such that the sinusoidal signal $s(k)$ is identical for all three. If one was to set, for instance, $f_s = 2f$, then it would be possible to obtain identical outputs $y(k) = s(t_k) = B$ for all time.

On the other hand, and more importantly, the result expressed in Theorem 1 is in complete agreement with the Nyquist-Shannon sampling theorem, in what concerns sufficiency. The necessity of the condition (9) will become clear in Section III-D. At this point, it is important to keep in mind that, given estimates of the system state $\hat{x}(k)$, one still needs to recover the signal parameters $A$, $B$, $\omega$, and $\phi$. Indeed, while the results that were derived so far allow to conclude that, as long as the sampling frequency satisfies (10), then the initial condition is uniquely determined, one still needs to consider how to recover the signal parameters from that initial condition.

C. Filter design

Before proceeding to the estimation of the actual sinusoidal signal parameters, it is important to detail how estimates of $\hat{x}(k)$ can be obtained. The structure of the system dynamics (6) is that of a discrete-time linear system. Therefore, linear theory applies and a multitude of observers for linear systems give simple solutions. In this paper the Kalman filter is selected, which gives globally exponentially stable error dynamics if the pair $(A(k), C)$ is uniformly completely observable. Uniform complete observability is a stronger form of observability that is nevertheless related to the first. In this case, uniform complete observability can be shown that under the same observability conditions.

D. Frequency estimates

As detailed in the previous sections, it is possible to design an observer for (6) such that, as long as (10) holds, the estimation error converges globally exponentially fast to zero. Since the system is observable, if an initial condition explains the system output, then that is indeed the initial condition, since it is uniquely determined. By construction, for (6) with $y(k) = s(t_k)$, the initial condition
\[
\begin{align*}
x_1(k_0) &= A \sin(\omega t_k + \phi) + B \\
x_2(k_0) &= 2A \sin \left( \frac{\omega t_k}{2} \right) \cos(\omega t_k + \phi + \frac{\omega T_s}{2}) \\
x_3(k_0) &= -4 \sin^2 \left( \frac{\omega t_k}{2} \right) - x_1(k) \\
x_4(k_0) &= 4B \sin^2 \left( \frac{\omega t_k}{2} \right) - x_4(k)
\end{align*}
\]
explains the system output for all $k \geq k_0$ and hence, since the state estimation error converges globally exponentially fast to zero, the estimates converge globally exponentially fast to the true values, which allows to write
\[
\lim_{k \to \infty} \left[ \begin{array}{c} x_1(k) \\
2A \sin \left( \frac{\omega t_k}{2} \right) \cos(\omega t_k + \phi + \frac{\omega T_s}{2}) - x_2(k) \\
-4 \sin^2 \left( \frac{\omega t_k}{2} \right) - x_3(k) \\
4B \sin^2 \left( \frac{\omega t_k}{2} \right) - x_4(k) \end{array} \right] = 0,
\]
where $\hat{x}_i(t)$ are the estimates of $x_i(t)$, $i = 1, 2, 3, 4$. However, notice that if the condition (9) is not imposed, there are in infinite number of solutions $(A, B, \omega, \phi)$ that satisfy (19) given the periodicity of the trigonometric functions. Hence, the necessity of condition (9).

Given estimates $\hat{x}(k)$, the next theorem addresses the issue of computing estimates of $\omega$.

**Theorem 2:** Consider an observer for (6) and let
\[
\hat{x}(k) := x(k) - \hat{x}(k) = \begin{bmatrix} x_1(k) \\
x_2(k) \\
x_3(k) \\
x_4(k) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(k) \\
\hat{x}_2(k) \\
\hat{x}_3(k) \\
\hat{x}_4(k) \end{bmatrix}
\]
denote its estimation error. Suppose that (9) holds and further assume that the origin of the error dynamics is a globally exponentially stable equilibrium point. Then:

i) there exists a discrete time instant $k_1$ such that
\[
-4 < \hat{x}_3(k) < 0
\]
for all $k \geq k_1$;

ii) if an estimate of $\omega$ is defined as
\[
\hat{\omega}(k) = \frac{2}{T_s} \arcsin \left( \sqrt{\frac{1}{4} \hat{x}_3(k)} \right) \in [0, \pi/T_s]
\]
for $k \geq k_1$, then the estimation error $\hat{\omega}(k) := \omega - \hat{\omega}(k)$ converges zero for all initial conditions; and

iii) in a neighborhood of $\hat{x} = 0$, the convergence of $\hat{\omega}(k)$ to zero is exponentially fast.
Proof: Since the estimation error converges to zero, it follows that
\[
\lim_{k \to \infty} \hat{x}_3(k) = \lim_{k \to \infty} x_3(k) - \hat{x}_3(k) = 0. \tag{22}
\]
Recalling the definition of \(x_3(k)\) in (4), which is actually constant, allows to write
\[
\lim_{k \to \infty} L - \hat{x}_3(k) = 0, \tag{23}
\]
with \(L := -4 \sin^2 \left( \frac{\omega T_s}{2} \right)\). Notice that, since (9) is true, it follows that (10) also holds and hence
\[-4 < L < 0. \tag{24}\]
From the definition of limit in (23), it follows that, for all \(\epsilon > 0\), it is possible to choose a discrete time instant \(k_1\) such that, for all \(k \geq k_1\), one has \(|L - \hat{x}_3(k)| < \epsilon\). In particular, for \(\epsilon_1 := \frac{L + 4}{2} > 0\), there exists a discrete time instant \(k_1\) such that, for all \(k \geq k_1\), one has \(|L - \hat{x}_3(k)| < \epsilon_1\). In particular, this implies that, for all \(k \geq k_1\), \(L - \hat{x}_3(k) < \epsilon_1\) or, equivalently,
\[
\hat{x}_3(k) > L - \epsilon_1 \iff \hat{x}_3(k) > \frac{L}{2} - 2. \tag{25}
\]
Using in (24) in (25) allows to conclude that, for all \(k \geq k_1\),
\[
\hat{x}_3(k) > -4. \tag{26}
\]
Now, for \(\epsilon_2 := -L/2 > 0\), there exists a second time instant \(K_{12}\) such that, for all \(k \geq k_{12}\), one has \(|L - \hat{x}_3(k)| < \epsilon_2\). In particular, this implies that, for all \(k \geq k_{12}\), \(L - \hat{x}_3(k) < -\epsilon_2\) or, equivalently,
\[
\hat{x}_3(k) < L + \epsilon_2 \iff \hat{x}_3(k) < \frac{L}{2}. \tag{27}
\]
Using (24) in (27) allows to conclude that, for all \(k \geq k_{12}\),
\[
\hat{x}_3(k) < 0. \tag{28}
\]
The proof of the first part of the theorem is concluded choosing \(k_t := \max(k_{11}, k_{12})\). Notice that, given that (20) holds for all \(k \geq k_t\), it follows that \(0 < \sqrt{-x_3(k)} < 1\) and hence \(\hat{\omega}(k)\) is well defined, with \(\hat{\omega}(k) \in [0, \pi/T_s]\). Now, from the definition of the estimation error, rewrite the frequency estimate as
\[
\hat{\omega}(k) = \frac{2}{T_s} \arcsin \left( -\frac{1}{4} \left[ x_3(k) - \hat{x}_3(k) \right] \right).
\]
Taking the limit and using (22) allows to conclude that
\[
\lim_{k \to \infty} \hat{\omega}(k) = \frac{2}{T_s} \arcsin \left( -\frac{1}{4} x_3(k) \right). \tag{29}
\]
The proof of the second part of the theorem is concluded using (4) in (29). To show the last part of the theorem, expand the error using (4) and (21) for \(k \geq k_t\), which gives
\[
|\hat{x}_3(k)| = |x_3(k) - \hat{x}_3(k)| = 4 \sin^2 \left( \frac{\omega T_s}{2} \right) - \sin^2 \left( \frac{\hat{\omega}(k) T_s}{2} \right) \tag{30}
\]
Using \(\sin^2 (\theta/2) = \frac{1 - \cos \theta}{2}\) twice in (30) gives
\[
|\hat{x}_3(k)| = 2 |\cos (\omega T_s) - \cos (\hat{\omega}(k) T_s)| \tag{31}
\]
and using \(\cos \theta - \cos \phi = -2 \sin \left( \frac{\theta + \phi}{2} \right) \sin \left( \frac{\theta - \phi}{2} \right)\) in (31) further gives
\[
|\hat{x}_3(k)| = 4 \left| \sin \left( \frac{\omega + \hat{\omega}(k) T_s}{2} \right) \sin \left( \frac{\omega - \hat{\omega}(k) T_s}{2} \right) \right|. \tag{32}
\]
Now, recall that, under the assumptions of the theorem, (10) holds and \(0 < \hat{\omega}(k) < \pi/T_s\). Thus, \(0 < \frac{\omega + \hat{\omega}(k) T_s}{2} < \pi\), which means that there exists a constant \(c_1 > 0\) such that
\[
\left| \sin \left( \frac{\omega + \hat{\omega}(k) T_s}{2} \right) \right| \geq c_1. \tag{33}
\]
Moreover, in a neighborhood of \(\hat{x} = 0\), to which corresponds a neighborhood of \(\hat{\omega} = 0\), there also exists a constant \(c_2 > 0\) such that
\[
\left| \sin \left( \frac{\omega - \hat{\omega}(k) T_s}{2} \right) \right| \geq c_2 |\hat{\omega}(k)|. \tag{34}
\]
Now, using (33) and (34) in (32) allows to conclude that there exists a constant \(c_0 > 0\) such that, in a neighborhood of \(\hat{x} = 0\),
\[
|\hat{x}_3(k)| \geq c_0 |\hat{\omega}(k)|. \tag{35}
\]
Finally, it is assumed that \(\hat{x}(k)\) converges exponentially fast to zero. Hence, there exist constant \(c > 0\) and \(\lambda > 0\) such that
\[
|x_3(k)| \leq ce^{-\lambda k}. \tag{36}
\]
The proof is concluding using (36) in (35), which gives
\[
|\hat{\omega}(k)| \leq \frac{c}{c_0} e^{-\lambda k}.
\]

E. Additional discussion

Although it is not the purpose of this paper, a brief discussion is presented here concerning the estimation of the offset. Notice that, from (4) and (5), it is possible to write
\[
B = \frac{x_3(k)}{x_3(k)}.
\]
Thus, as long as estimates \(\hat{x}_3(k) < 0\) and \(\hat{x}_4(k) > 0\) are available, an estimate of the offset is given by
\[
\hat{B}(k) = \frac{\hat{x}_4(k)}{\hat{x}_3(k)}.
\]
Once offset and frequency estimates are available, different methods in the literature can be applied to determine the amplitude \(A\) and the phase \(\phi\). That is out of the scope of this paper.

IV. Simulation results

Simulation results are presented and discussed in this section to illustrate the proposed method for frequency estimation of a sinusoidal signal. First, a constant frequency sinusoid is assumed, in Section IV-A, whereas a chirp signal is considered in Section IV-B.
A. Constant frequency

The convergence and performance of the proposed method to estimate the frequency of a sinusoidal signal is illustrated in this section. The sinusoidal signal that is considered is

\[ s(t) = \sin(2\pi t) + 1 \]

which corresponds to a sinusoid with amplitude \( A = 1 \), frequency \( f = 100 \) Hz, phase \( \phi = 0 \) rad, and offset \( B = 1 \). This signal was sampled with a sampling frequency \( f_s = 1000 \) Hz, hence \( f_s > f \). Moreover, zero mean Gaussian noise, with standard deviation \( 10^{-2} \), was added to the measurements. To estimate the frequency and offset of the signal, a Kalman filter was implemented, as described in Section III-C. The state disturbance covariance matrix was set to \( 10^{-6} \), whereas the output noise covariance matrix was chosen as \( 10^{-2} \). The former was chosen empirically, whereas the latter was set according to the standard deviation of the measurements noise. The initial estimate was set to zero.

The initial convergence of the frequency estimation error is depicted in Fig. 1, whereas the steady-state evolution is shown in Fig. 2. From the first plot it is evident that the filter converges very quickly, in less than 0.01 s, which corresponds to the period of the signal. The steady-state performance is also very good, with the error 0.2 Hz for most of the time. As a byproduct, the offset of the signal was also computed as detailed in Section III-E. The initial convergence of the offset is shown in Fig. 3, whereas the steady-state behavior is depicted in Fig. 4. The same convergence rate can be observed and the offset error remains, in steady-state, below 0.05 most of the time, which compares to the standard deviation of the noise of 0.01.

B. Chirp signal

The frequency of a signal may change over time. In this section, the robustness of the proposed method is illustrated considering a linear chirp signal. In particular, the instantaneous frequency of the signal is given by \( f(t) = t \) (Hz) or, equivalently, the rate of change of the frequency is 1 Hz/s. All the conditions of the simulation described in the previous section were maintained, namely the characteristics of the additive Gaussian noise and the parameters and initial conditions of the Kalman filter.

The initial convergence of the frequency estimation error is depicted in Fig. 5, whereas the steady-state evolution is shown in Fig. 6. The first obvious difference is that the filter takes a longer time to converge. This is related not only to the changing frequency of the signal but also to the lower initial frequencies of the signal, which may lead to larger convergence times. The second obvious difference concerns the steady-state behavior. Naturally, the mean error is no
longer zero, as the frequency of the signal is not constant, changing at a rate of 1 Hz/s. Interestingly enough, the mean error is about 1 Hz. Nevertheless, this also evidences good performance. Indeed, at \( t = 180 \) s, the frequency of the signal is 180 Hz, and it is estimated with an error of about 1 Hz.

Again as a byproduct, the offset of the signal was also computed as detailed in Section III-E. The initial convergence of the offset is shown in Fig. 7, whereas the steady-state behavior is depicted in Fig. 8. The same convergence rate can be observed and the offset error remains, in steady-state, below 0.01 most of the time, which compares to the standard deviation of the noise of 0.01.

**V. Conclusions**

The problem of estimation of the unknown constant parameters of a biased sinusoidal signal has long received the attention of the research community, as it is vital to the solution in a number of areas. This paper proposes a novel estimator for all unknowns featuring globally exponentially stability (GES), in the discrete time case, given the samples available. Simulations results were presented that illustrate the achievable performance. Future work will include extensive Monte Carlo simulations and the comparison with Cramer-Rao lower bounds in several cases of interest as well as the experimental evaluation in realistic scenarios.

**REFERENCES**