Nonlinear Attitude Observer on $SO(3)$ Based on Single Body-Vector Measurements

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Abstract—This paper proposes a nonlinear attitude observer, built on the special orthogonal group $SO(3)$, that resorts to body measurements of a single constant inertial vector, in addition to the angular velocity obtained from a set of three high-grade fiber optic gyros, which are herein assumed sensitive enough to measure the Earth’s rotation vector. This strategy contrasts with typical attitude solutions that require measurements of two non-collinear inertial vectors, and can be very useful in scenarios with either strong magnetic anomalies or highly-accelerated vehicles, where either the magnetometer or accelerometer measurements may no longer be useful for attitude estimation in traditional solutions. The attitude estimation solution features a tuning parameter, in the form of a positive scalar gain, and the nonlinear error dynamics associated to the observer are shown to be locally exponentially stable. Simulation results with realistic noise are presented that allow to assess the performance of the proposed solution.

Index Terms—Attitude Observer, Single Body-Vector, Topological Constraints, Local Exponential Stability

I. INTRODUCTION

The development of algorithms for estimating the attitude of rigid bodies remains an ubiquitous research goal across several engineering fields. Indeed, due to the myriad of applications which considerably benefit from their implementation, these algorithms are still actively researched and pose compelling theoretical and practical challenges.

Wahba’s seminal work [1] proposed, for the first time, an optimality based approach to solve the problem of attitude determination by minimizing a best least squares loss function. The result, which can be obtained, for instance, from a singular-value decomposition [2], yields a proper orthogonal matrix that better maps two sets of $n$ ($\geq 2$) vectors onto each other. This problem has since then become a reference within the area of spacecraft attitude estimation, for which several contributions can be found in the literature. These are typically divided into two groups: nonlinear observers - the reader is referred to the survey [3] and references therein - and, Kalman filter based estimators - see, for example, [4] and [5]. It is worth noting that some observer based strategies have often been extended to control schemes, as studied in [6], [7], and [8].

Spacecraft attitude estimators typically concern low-Earth orbit applications, where known inertial reference vectors are usually time-varying, thus offering an intrinsic layer of additional spacial information. Combining readings of known inertial vectors with their measured counterparts allows for the design of globally exponentially stable attitude observers [9]. Particularly, in [8], a discrete-time semi-global practical asymptotic observer is presented that relies on single magnetometer readings, although, due to the nature of the observer proposed therein, topological properties are not preserved. An alternative can be found in [10], where, in a continuous-time setting, while also using magnetometer measurements, the estimation of the attitude of a spacecraft occurs in simultaneous with the estimation of an $N$-vector of bias parameters. The solution derived resorts to the use of a probability distribution function defined on the Cartesian product of the special orthogonal group $SO(3)$, the group of rotation matrices, and the Euclidean space $\mathbb{R}^N$.

This notwithstanding, recently, the fast development of mobile platforms and unmanned vehicles, in addition to performance improvements over low-cost sensors, motivated the design of a new series of attitude estimators that explore the combination of measurements from different sensors. The work in [11] studies the design of a nonlinear complementary filter on $SO(3)$ based directly on inertial measurements for applications in control of unmanned aerial vehicles. Similarly, in [12] an attitude and gyro-bias estimator is developed that addresses the problem of weak dynamics. Other works have extended this working paradigm to fuse visual data with GPS information in order to obtain an estimate of the attitude matrix [13].

In this work, a most challenging theoretical approach is examined: to use only one constant inertial vector as reference. Furthermore, the Earth’s rotation is taken into account when designing the observer. The inclusion of the Earth’s angular velocity in the development of attitude estimators has been pursued in [14], as well as in past work by the authors, see [15] and [16]. The work presented in this paper builds upon the results achieved in [16], where an observer evolving on the 2-sphere manifold is designed and shown to be semiglobally stable. In that paper, a cascade of observers is proposed that, in a first stage, yields estimates of an auxiliary vector which is closely related to the Earth’s angular motion. The resulting estimates, combined with measurements of a constant inertial vector, are then fed to a second observer that produces estimates of the corresponding rotation matrix.

In this paper, a single observer is proposed, built on $SO(3)$, that relies exclusively on measurements of one
constant inertial vector, hence discarding the usage of a second auxiliary vector. The nonlinear observer is shown to be locally exponentially stable.

The rest of the paper is structured as follows: in Section II a brief description of the problem statement is presented and a nonlinear attitude observer is designed that leads to the main result of this work. In Section III simulation results are shown to illustrate the performance of the proposed solution along with some pertinent discussions. Finally, Section IV contains a few conclusions in addition to hints for future work.

A. Notation

Throughout the paper, a bold symbol stands for a multidimensional variable, the symbol 0 denotes a matrix of zeros and I an identity matrix, both of appropriate dimensions. The Special Orthogonal Group is denoted by $SO(3) := \{ X \in \mathbb{R}^{3 \times 3} : XX^T = X^T X = I \land \det(X) = 1 \}$. In $\mathbb{R}^3$, the skew-symmetric matrix of a generic vector $a \in \mathbb{R}^3$, with $a = [a_x, a_y, a_z]^T$, is defined as $S[a]$, such that given another generic vector $b \in \mathbb{R}^3$ one has $a \times b = S[a]b$, where

$$S[a] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$  

Finally, for convenience, the transpose operator is denoted by the superscript $(\cdot)^T$.

II. DESIGN OF ATTITUDE OBSERVER

This section includes a brief introduction to the problem statement, followed by the steps in designing an attitude observer that resorts to single measurements of one constant inertial vector.

A. Problem Statement

Consider a robotic platform with a body-fixed frame associated to it. Suppose this platform is equipped with a set of three, high-grade, orthogonally mounted gyros, for instance the novel KVH® DSP-1760 3-axis Fiber Optic Gyroscope, which provides measurements of angular velocity. Further suppose that there is a second sensor measuring a body vector whose inertial counterpart is constant. Accelerometers and magnetometers are apt examples of such sensors, although one must be careful ensuring that their functioning is immune to external electrical and/or mechanical disturbances. Moreover, assume that the gyros are sensitive to the Earth’s rotation, which means that the angular velocity readings feature implicit measurements of said rotation. Overall, all measurements with respect to the body-fixed frame can be expressed in a given inertial frame by means of an unknown rotation matrix. Hence, the aim of this paper is to present an observer for the orientation matrix based on the angular velocity readings and on measurements of the constant inertial vector. As opposed to previous work by the authors, and to most solutions found in the literature, the observer presented herein resorts solely to a single measured vector while simultaneously preserving topological properties. This trait, albeit a remarkable achievement in terms of simplified setup design, exacerbates the structural complexity of the observer and, in turn, the stability analysis of the problem.

B. Observer for the orientation matrix

Let $R(t) \in SO(3)$ denote the rotation matrix from a body-fixed frame $\{B\}$ to a local inertial coordinate reference frame $\{I\}$. The evolution in time of this rotation matrix obeys

$$\dot{R}(t) = R(t)S[\omega(t)],$$  

where $\omega(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$ with respect to $\{I\}$, as expressed in $\{B\}$. The measurements from the set of three high-grade, orthogonally mounted rate gyros are given by

$$\omega_m(t) = \omega(t) + \omega_E(t),$$  

where $\omega_E(t) \in \mathbb{R}^3$ is the angular velocity of the Earth around its own axis, as expressed in $\{B\}$. In turn, let the vector measurements provided by the second sensor be denoted as $m(t) \in \mathbb{R}^3$. These, when expressed in inertial coordinates, are assumed to be constant. Hence, let $\hat{\omega}_E$ and $\hat{m}$ be the inertial vector counterparts corresponding to $\omega_E(t)$ and $m(t)$, respectively, such that $\hat{\omega}_E = R(t)\omega_E(t)$ and $\hat{m} = R(t)\hat{m}(t)$ for all $t > 0$. For ease of notation, the upper left superscripts of body vectors were dropped, and thus $B\omega \equiv \omega_E$. From [2], equation (1) can be rewritten as

$$\dot{R}(t) = R(t)S[\omega_m(t) - \omega_E(t)].$$

The following assumptions are considered throughout the remainder of the paper.

Assumption 1. The constant inertial vectors $\hat{\omega}_E$ and $\hat{m}$ are not collinear, i.e., $\hat{\omega}_E \times \hat{m} \neq 0$.

This assumption concerns observability purposes and is easily attainable in practical terms. In other words, it ensures that one can extract unequivocal information on directionality from the two vectors involved as long as they define a plane in space.

Assumption 2. The rate gyro measurements are bounded for all time, i.e., there exists a $\sigma > 0$ such that, for all $t > 0$, $\|\omega_m(t)\| \leq \sigma$.

Consider now the following observer for the rotation matrix

$$\dot{\hat{R}}(t) = \hat{R}(t)S[\omega_m(t) - \hat{R}^T(t)\hat{\omega}_E + \alpha m(t) \times (\hat{R}^T(t)\hat{m})],$$  

where $\alpha \in \mathbb{R}$ is a constant positive gain to be determined. Define also the error variable

$$\tilde{R}(t) := R(t)\hat{R}(t) \in SO(3),$$

whose dynamics are given by
\[ \dot{\mathbf{R}}(t) = \mathbf{R}(t) \dot{\mathbf{R}}^T(t) + \mathbf{R}(t) \dot{\mathbf{R}}^T(t) \]

\[ = \mathbf{R}(t) \mathbf{S} \left[ \omega_m(t) - \mathbf{E}(t) \right] \dot{\mathbf{R}}^T(t) + \mathbf{R}(t) \left\{ \dot{\mathbf{R}}(t) \mathbf{S} \left[ \omega_m(t) - \mathbf{E}(t) \right] \right\}^T \]

\[ - \dot{\mathbf{R}}^T(t) \dot{\mathbf{E}} + \alpha m \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \]

\[ = \mathbf{R}(t) \mathbf{S} \left[ \omega_m(t) - \mathbf{E}(t) \right] \dot{\mathbf{R}}^T(t) - \mathbf{R}(t) \mathbf{S} \left[ \omega_m(t) - \mathbf{E}(t) \right] \dot{\mathbf{R}}^T(t) \]

Isolating the terms associated with the Earth’s angular velocity, and further noticing that the terms corresponding to the measurements of angular velocity cancel each other, allows to write

\[ \dot{\mathbf{R}}(t) = - \mathbf{R}(t) \mathbf{S} \left[ \omega_E(t) - \dot{\mathbf{R}}^T(t) \dot{\mathbf{E}} \right] \dot{\mathbf{R}}^T(t) \]

\[ - \mathbf{R}(t) \mathbf{S} \left[ \alpha m \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \right] \dot{\mathbf{R}}^T(t) \]

\[ = - \mathbf{R}(t) \mathbf{S} \left[ \left( \dot{\mathbf{R}}^T - \mathbf{R}(t) \right) \dot{\mathbf{E}} \right] \dot{\mathbf{R}}^T(t) \]

\[ - \mathbf{R}(t) \mathbf{S} \left[ \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \right] \dot{\mathbf{R}}^T(t). \]

Since \( \mathbf{R}(t)^T \mathbf{R}(t) = \mathbf{I} \), rewrite the previous result as

\[ \dot{\mathbf{R}}(t) = - \mathbf{R}(t) \mathbf{S} \left[ \left( \dot{\mathbf{R}}^T - \mathbf{R}(t) \right) \dot{\mathbf{E}} \right] \dot{\mathbf{R}}^T(t) \]

\[ - \mathbf{R}(t) \mathbf{S} \left[ \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \right] \dot{\mathbf{R}}^T(t). \]

Recall the error definition in (4), and employ the property

\[ \mathbf{R}(t) \mathbf{S} [\mathbf{a}] \mathbf{R}(t) = \mathbf{S} [\mathbf{R}(t) \mathbf{a}], \quad \mathbf{a} \in \mathbb{R}^3, \]

to help simplifying equation (5) as

\[ \dot{\mathbf{R}}(t) = - \mathbf{R}(t) \mathbf{S} \left[ \left( \dot{\mathbf{R}}^T - \mathbf{R}(t) \right) \dot{\mathbf{E}} \right] \dot{\mathbf{R}}^T(t) \]

\[ - \mathbf{R}(t) \mathbf{S} \left[ \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \right] \dot{\mathbf{R}}^T(t) \]

\[ = - \mathbf{R}(t) \mathbf{S} \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \dot{\mathbf{R}}^T(t) \]

\[ = - \mathbf{R}(t) \mathbf{S} \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \dot{\mathbf{R}}^T(t). \]

Before proceeding to the main result of this work, let us consider a linear approximation for the system expressed by (6). In practical terms, assume that only small perturbations of the rotation matrix exist.

First, let \( \mathbf{\theta}(t) \in \mathbb{R}^3 \) denote the Euler angles associated with \( \mathbf{R}(t) \) and let \( \mathbf{u} \in \mathbb{R}^3 \) be a constant arbitrary vector. Second, consider the estimated rotation matrix \( \hat{\mathbf{R}}(t) \) as the result of a slight perturbation over the nominal rotation matrix, and regard it as a parameterization of \( \mathbf{R}(t) \) in terms of the nominal Euler angles \( \mathbf{\theta} \) and of an infinitesimal deviation denoted by \( \delta \mathbf{\theta} \). Hence, the Taylor-series expansion of \( \hat{\mathbf{R}}(t) \mathbf{u} = \mathbf{R} \left( \mathbf{\theta} + \delta \mathbf{\theta} \right)(t) \mathbf{u} \) can be written as

\[ \hat{\mathbf{R}}(t) \mathbf{u} = \mathbf{R} \left( \mathbf{\theta} + \delta \mathbf{\theta} \right)(t) \mathbf{u} + \frac{\partial \left( \mathbf{R} \left( \mathbf{\theta} \right) \mathbf{u} \right)}{\partial \mathbf{\theta}} \bigg{|}_{\mathbf{\theta}} \delta \mathbf{\theta}(t) + \text{h.o.t.} \]

which, after applying a first order approximation, results in (vide [17])

\[ \hat{\mathbf{R}}(t) \approx \left( \mathbf{I} - \mathbf{S} \left[ \delta \times \delta \mathbf{\theta} \right] \right) \mathbf{R}(t), \]

where, for the sake of readability, \( \mathbf{R}(t) \equiv \mathbf{R} \left( \mathbf{\theta} \right)(t) \). Therefore, according to (4), it follows that

\[ \hat{\mathbf{R}}(t) = \mathbf{I} + \mathbf{S}[x(t)], \]

where \( x(t) := \left( \delta \times \delta \mathbf{\theta} \right) \) is a pseudo rotation vector whose components correspond to infinitesimal rotations about the three axes of the reference frame.

Now, according to the linearization stated in (7), substitute in (6) all terms expressed by \( \hat{\mathbf{R}}(t) \) and simplify in order to get

\[ \dot{\mathbf{R}}(t) = - \mathbf{S} \left[ \left( \mathbf{I} + \mathbf{S}[x(t)] \right) \dot{\mathbf{E}} \right] \dot{\mathbf{R}}^T(t) \]

\[ - \mathbf{S} \left[ \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \right] \dot{\mathbf{R}}^T(t) \]

\[ = - \mathbf{S} \left[ \left( \mathbf{I} + \mathbf{S}[x(t)] \right) \dot{\mathbf{E}} \right] \dot{\mathbf{R}}^T(t) \]

\[ - \mathbf{S} \left[ \alpha \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \times \left( \dot{\mathbf{R}}^T(t) \dot{m} \right) \right] \dot{\mathbf{R}}^T(t). \]

Recall that \( \mathbf{S}[x(t)] \) is a locally asymptotically stable equilibrium point and, accordingly, the attitude estimates provided by (7) converge locally asymptotically fast to the actual values.

Theorem 1. Consider the nonlinear attitude observer (3), the dynamics of the rotation matrix error as given by (6), and the LTI system (5). Suppose that \( \alpha > 0 \) and that Assumptions 1 and 2 hold. Then the origin of the nonlinear error dynamics (9) is a locally asymptotically stable equilibrium point and, accordingly, the attitude estimates provided by (6) converge locally asymptotically fast to the actual values.

Proof. Finding, analytically and in function of the parameter \( \alpha \), the eigenvalues of the matrix \( \mathbf{A}(\alpha) \), which is not symmetric and is extremely ill-conditioned, is not a straightforward task. There are a few methods to infer about the stability of an LTI system, such as the Routh–Hurwitz stability criterion, which encloses a necessary and sufficient condition; this criterion is tested below.

Start by considering the third degree characteristic polynomial associated with the eigenvalues \( \lambda \in \mathbb{C} \) of the matrix \( \mathbf{A}(\alpha) \) given by

\[ a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0, \]

where \( a_0, a_1, a_2, a_3 \in \mathbb{R} \).

Next, build the corresponding Routh-Hurwitz table, which is given in Table 1.
TABLE I: Generic Routh-Hurwitz table for 3rd order characteristic polynomial.

After tedious, long, but straightforward computations, it is possible to write the columns of Table I as shown in Table II.

<table>
<thead>
<tr>
<th>$\alpha^2|\mathbf{I}|^4 + |\mathbf{I}|\mathbf{I}|^2$</th>
<th>$\alpha|\mathbf{I}|\mathbf{I}|^2 - |\mathbf{I}|\mathbf{I}|^2$</th>
<th>$\alpha|\mathbf{I}|\mathbf{I}|\mathbf{I}|^2$</th>
<th>$\alpha|\mathbf{I}|\mathbf{I}|\mathbf{I}|^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2$\alpha|\mathbf{I}|^2$</td>
<td>$2|\mathbf{I}|^2\left(\alpha^2|\mathbf{I}|^2 + |\mathbf{I}|\mathbf{I}|^2\right) - |\mathbf{I}|\mathbf{I}|^2$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

TABLE II: Routh-Hurwitz table associated with matrix $\mathbf{A}(\alpha)$.

If there are no sign changes along the first column (pivot column) of the Routh-Hurwitz table, Table II, and none of its elements is zero, then the roots of (9) are strictly negative. Hence, one immediately deduces that, under the established assumptions, both $2\alpha\|\mathbf{I}\|^2$ and $\alpha\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|^2$ are always positive. Finally, it is left to verify that

$$2\|\mathbf{I}\|^2 \left(\alpha^2\|\mathbf{I}\|^2 + \|\mathbf{I}\|\mathbf{I}\|^2\right) - \|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|^2 > 0,$$

which is quite obvious since

$$\|\mathbf{I}\|^2 \|\mathbf{I}\|\mathbf{I}\|^2 \geq \|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|^2.$$

As result, it is implied that all the eigenvalues of $\mathbf{A}(\alpha)$ have negative real part, i.e., that the LTI system is stable.

Then, by invoking [18, Theorem 4.7], one can conclude that the equilibrium point $x = 0$ for the LTI system is also an equilibrium point for the original nonlinear system (5), which means that $\mathbf{R}(t) \to \mathbf{I}$ when $t \to \infty$, or, likewise, that $\mathbf{R}(t) \to \mathbf{R}(t)$ when $t \to \infty$. Therefore, the nonlinear observer (3) is locally asymptotically stable, thus concluding the proof.

III. SIMULATION RESULTS

This section presents the results obtained from a set of simulations that illustrate the achievable performance of the proposed nonlinear attitude observer.

Consider a robotic platform describing a rotational motion in a three-dimensional space, located at a latitude of $\varphi = 38.777816^\circ$, a longitude of $\psi = 9.097570^\circ$, and at sea level. Taking into account the length of time known as sidereal day, the corresponding norm of the Earth’s angular velocity is $\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|^2 = 7.2921159 \times 10^{-5}$ rad/s, whose vectorial representation in the NED frame is given by

$$\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|\mathbf{I}\|^2 = 7.2921159 \times 10^{-5} \text{ rad/s}.$$
dynamics of the LTI system $\tilde{\theta}$, which are dominated by the rather slow Earth’s angular rotation. Notice also that the dynamics are independent of the trajectory of the platform. The corresponding evolution of the angle error is shown in Figure 2. After roughly 12 hours, the angle error enters in a steady-state. The mean value and the standard deviation were computed for $t > 15$ hours, resulting in 0.1417 degrees and 0.05632 degrees, respectively, which are good results for this kind of application.

For the sake of completeness, Figure 3 depicts the evolution of all entries of $\tilde{\mathbf{R}}(t)$ as given by (4). It is obvious that this proper orthogonal matrix is converging to an identity, which means the observer estimates are converging to the true values.

The observer was further tested for different initial conditions, up to values close to the 180-degree maximum deviation, with the performance, in terms of convergence time, consistently degrading significantly for larger perturbations, despite the estimates converging to the real values. In some cases, 24 hours were not enough for the observer to even reach steady state, as evidenced in Figures 4 and 5, which were made with respect to an initial deviation of $\tilde{\theta}(0) = 175$ degrees.

IV. CONCLUSIONS

In this paper, a nonlinear attitude observer built on $SO(3)$ was proposed that takes into account the Earth’s rotation and resorts exclusively to single measurements of a constant inertial vector, in addition to angular velocity readings. Due to the highly nonlinear structure of the attitude error dynamics, a linearization was carried out that resulted in an LTI system, which was then shown to be asymptotically stable. Therefore, the proposed observer is better characterized as locally exponentially stable. In a simulation environment with realistic noise, the performance of the attitude estimator was shown to be slow but compatible with high-grade attitude determination systems, which take a long time to converge but exhibit very good performance. Future work will include the study of modifications of the proposed observer to im-

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Fig. 1: Evolution in time of $x(t)$ for $\tilde{\theta}(0) = 10$ degrees.

Fig. 2: Angle error evolution for $\tilde{\theta}(0) = 10$ degrees.

Fig. 3: Evolution of $\mathbf{R}$ entries for $\tilde{\theta}(0) = 10$ degrees. Upper right corner: diagonal entries of $\mathbf{R}$.

Fig. 4: Angle error evolution for $\tilde{\theta}(0) = 175$ degrees.

Fig. 5: Evolution in time of $x(t)$ for $\tilde{\theta}(0) = 175$ degrees.
prove the convergence rate while maintaining the topological characteristics of the problem.

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