Position and Velocity Filters for Intervention AUVs based on Single Range and Depth Measurements

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Abstract—This paper proposes novel cooperative navigation solutions for an Intervention Autonomous Underwater Vehicle (I-AUV) working in tandem with an Autonomous Surface Craft (ASC). The I-AUV is assumed to be moving in the presence of constant unknown ocean currents, and aims to estimate its position relying on measurements of its range to the ASC and of its depth relatively to the sea level. Two different scenarios are considered: in one, the ASC transmits its position and velocity to the I-AUV, while in the other the ASC transmits only its position, and the I-AUV has access to measurements of its velocity relative to the ASC. A sufficient condition for observability and a method for designing state observers with Globally Asymptotically Stable (GAS) error dynamics are presented for both problems. Finally, simulation results are included and discussed to assess the performance of the proposed solutions in the presence of measurement noise.

I. INTRODUCTION

In the past decades, navigation and control of autonomous vehicles has been the subject of intensive study, fostered by the myriad of potential applications they offer. One in particular, the development of Autonomous Underwater Vehicles (AUVs) with intervention capabilities, would provide a valuable asset to fields as diverse as marine rescue, scientific missions, and offshore industries, among many others. In fact, most current applications use manned submersibles or remotely operated vehicles (ROVs), as the high level of autonomy and perception required for autonomous intervention has not yet been reached. However, both of these alternatives pose strong drawbacks. The use of manned submersibles requires putting humans in harm’s way, and also implies limited autonomy at a high financial cost, due to the use of a support oceanographic vessel. For ROVs operation, although danger to the pilot is not a factor, their operation over time will still be limited by fatigue of the human operators. Besides that, the use of a ROV will also usually require, in addition to the support vessel, an automatic Tether Management System (TMS) and a Dynamic Position (DP) system. In this context, the goal of the EU project TRIDENT is to design a team of two autonomous vehicles with complementary capabilities, an Autonomous Surface Craft (ASC) and an Intervention Autonomous Underwater Vehicle (I-AUV), equipped with a dexterous manipulator, used to carry out underwater manipulation tasks. This paper proposes a novel navigation solution for the I-AUV based on single range readings to the ASC and depth measurements, aided by auxiliary sensors to provide awareness of its movement, and communication between the two robots.

Over the last few years, several approaches to the problem have been studied, see [9] for a recent, comprehensive survey on the field. One of those is Dead Reckoning [10], whose performance is very good in the short-term but necessarily degrades over time. Another solution is localization aided by artificial beacons that can be disposed in several configurations such as Long Base Line (LBL), Short Base Line (SBL), and Ultra-Short Base Line (USBL), see e.g. [6], [7], [11], and [13]. Although this approach yields good long-term results, it poses several operational constraints that may be overwhelming in many applications. In fact, the beacons must be previously deployed in the planned area of operation, and their position must be carefully and precisely calibrated. More recently, cooperative navigation solutions based on range measurements have been studied, such as in [1], [2], and [15]. This paper presents an alternative solution, in which the navigation system onboard the I-AUV is designed with respect to the ASC. Assuming the ASC has its own navigation system (see e.g. [14] for an experimentally validated navigation solution for the DELFIMx, an ASC associated with project TRIDENT), communication with the I-AUV will allow the latter to recover its absolute position.

This paper addresses the problem of cooperative navigation of an I-AUV/ASC tandem, in which the I-AUV estimates its position relying on measurements of its range to the ASC and of its depth relatively to the sea level, aided by auxiliary sensors and transmission of relevant data between the ASC and the I-AUV. Two scenarios are considered: in the first, the ASC transmits both its position and velocity to the I-AUV, which is assumed to be moving in the presence of constant unknown drifts. It is assumed that the I-AUV has access to a measurement of its velocity relative to the water,
Given e.g. by a Doppler Velocity Log (DVL). In the second scenario, the ASC transmits only its position to the I-AUV, which is assumed to have access to readings of its velocity relative to the ASC, given e.g. by an Acoustic Vector-Sensor Array (AVSA) [12]. The dynamics of the problem are derived from the linear motion kinematics of vehicles moving in 3-D, which are exact, and nonlinear space-state representations are presented for both cases. Considering this, and even though the present work is strongly motivated by the case of an I-AUV/ASC tandem in the scope of project TRIDENT, the results presented here can be extended to other classes of vehicles and/or sensors. The observability of both nonlinear systems is studied using state augmentation. By carefully defining new state variables, it is possible to derive Linear Time-Varying (LTV) systems which mimic exactly the corresponding nonlinear ones. Classical estimation and filtering theory is then applied to the LTV systems, allowing for the derivation of sufficient conditions for observability for both cases. As the observability analysis is carried out on linear systems, it allows for the design of state observers with estimation errors whose dynamics are guaranteed to be Globally Asymptotically Stable (GAS), which would not be possible using classical methods such as the Extended Kalman Filter (EKF). Following this, a Kalman filter for the LTV systems with GAS error dynamics is proposed, and its performance is assessed in simulation. Previous work by the authors on the subject of linear motion estimation based on range measurements can be found in [3], where the case with a stationary source was considered. The present paper extends those results for a mobile source and also provides enhanced proofs. The addition of a depth reading allows for operation of the I-AUV at constant depth, which may be useful or even critical in certain applications.

The paper is organized as follows: Section II introduces the problem dynamics, while Section III details the observability analysis of the two nonlinear systems. Section IV presents simulation results to assess the performance of the proposed solution and, finally, Section V summarizes the main conclusions of the paper.

A. Notation

Throughout the paper the symbol 0 denotes a matrix (or vector) of zeros and I an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as \( \text{diag}(A_1, \ldots, A_n) \).

II. PROBLEM STATEMENT

Consider an I-AUV moving underwater in the presence of constant unknown ocean currents and working in tandem with an ASC. It is assumed that the ASC has a built-in navigation system which provides accurate estimates of both its inertial position and velocity, and can communicate them to the I-AUV (using, e.g., an acoustic modem). Suppose that the I-AUV has access to a measurement of its distance, or range, \( r(t) \in \mathbb{R} \) to the ASC, and to a measurement of its depth relative to the sea level, \( d(t) \in \mathbb{R} \). The problem considered in this paper is that of estimating the position of the ASC relative to the I-AUV using the range and depth sensors as navigation aiding devices.

Let \( \{I\} \) denote an inertial reference coordinate frame and \( \{B\} \) a coordinate frame attached to the I-AUV, denominated in the sequel as the body-fixed coordinate frame. The linear motion of the I-AUV can be written as

\[
\mathbf{p}(t) = \mathbf{R}(t)\mathbf{v}(t),
\]

where \( \mathbf{p}(t) \in \mathbb{R}^3 \) denotes the inertial position of the I-AUV, \( \mathbf{v}(t) \in \mathbb{R}^3 \) is the velocity of the I-AUV relative to \( \{I\} \) and expressed in body-fixed coordinates, and \( \mathbf{R}(t) \) is the rotation matrix from \( \{B\} \) to \( \{I\} \), which verifies \( \dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}(\omega(t)) \), where \( \omega(t) \in \mathbb{R}^3 \) is the angular velocity of the I-AUV, expressed in body-fixed coordinates, and \( \mathbf{S}(x) \in \mathbb{R}^{3 \times 3} \) is the skew-symmetric matrix such that \( \mathbf{S}(x)y \) is the cross product \( x \times y \). Let \( s(t) \in \mathbb{R}^3 \) denote the inertial position of the ASC, and \( \mathbf{v}_s(t) \in \mathbb{R}^3 \) its inertial velocity. Then, the range to the ASC is given by \( r(t) = ||\mathbf{r}(t)|| \), where

\[
\mathbf{r}(t) = \mathbf{R}(t)\mathbf{s}(t) - \mathbf{p}(t) \in \mathbb{R}^3
\]

is the position of the ASC relative to the I-AUV, expressed in body-fixed coordinates, precisely the quantity that the I-AUV aims to estimate. The time derivative of (1) follows

\[
\dot{\mathbf{r}}(t) = -\mathbf{S}(\omega(t))\mathbf{r}(t) + \mathbf{R}(t)\mathbf{v}_s(t) - \mathbf{v}(t).
\]

It is assumed that the sensor suite on-board the I-AUV provides measurements of its attitude and angular velocity, \( \mathbf{R}(t) \) and \( \omega(t) \). As the I-AUV is evolving in the presence of unknown currents, it might not be possible to measure its inertial velocity directly, for example if the I-AUV is moving far away from the seabed. However, its velocity relative to the fluid is available, as measured by a DVL. Regarding the velocity of the ASC, two cases will be considered. In the first case, the ASC communicates its velocity, \( \mathbf{v}_s(t) \), to the I-AUV, and in the second case, the I-AUV has access to measurements of its velocity relative to the ASC, \( \Delta\mathbf{v}(t) := \mathbf{R}(t)\mathbf{v}_s(t) - \mathbf{v}(t) \), provided for instance by an AVSA.

For the first case, let \( \mathbf{v}_r(t) \in \mathbb{R}^3 \) and \( \mathbf{v}_f(t) \in \mathbb{R}^3 \) denote the velocity of the I-AUV relative to the fluid and the velocity of the fluid relative to \( \{I\} \), respectively, both expressed in body-fixed coordinates. Considering that the velocity of the fluid is constant, it is possible to further write

\[
\begin{aligned}
\dot{\mathbf{r}}(t) &= -\mathbf{S}(\omega(t))\mathbf{r}(t) + \mathbf{R}(t)\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_f(t), \\
\dot{\mathbf{v}}_f(t) &= -\mathbf{S}(\omega(t))\mathbf{v}_f(t).
\end{aligned}
\]

Now, let

\[
\begin{bmatrix}
\mathbf{x}_1(t) \\
\mathbf{x}_2(t)
\end{bmatrix} :=
\begin{bmatrix}
\mathbf{R}(t) & 0 \\
0 & -\mathbf{R}(t)
\end{bmatrix}
\begin{bmatrix}
\mathbf{r}(t) \\
\mathbf{v}_f(t)
\end{bmatrix}
\]

\[
\mathbf{y}(t) :=
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} =
\begin{bmatrix}
\mathbf{r}(t) \\
d(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & \mathbf{x}_1(t)
\end{bmatrix},
\]

which is a Lyapunov coordinate transformation, and thus preserves stability and observability properties [5], and define \( \mathbf{u}_1(t) := \mathbf{v}_s(t) - \mathbf{R}(t)\mathbf{v}_r(t) \). Computing the time derivatives of \( \mathbf{x}_1(t) \) and \( \mathbf{x}_2(t) \) gives the nonlinear system

\[
\begin{aligned}
\dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) + \mathbf{u}_1(t), \\
\dot{\mathbf{x}}_2(t) &= 0, \\
\mathbf{y}(t) &= \begin{bmatrix} 0 & 0 & 1 & \mathbf{x}_1(t) \end{bmatrix},
\end{aligned}
\]

(2)
where $x_1(t), x_2(t) \in \mathbb{R}^3$ are the system states, $u_1(t) \in \mathbb{R}^3$ is the system input, and $y(t) \in \mathbb{R}^2$ represents the system output. Let

$$
\begin{align*}
&x_1(t) := \mathbf{R}(t)r(t) \\
y(t) := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} \|x_1(t)\| \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_1(t) \end{bmatrix},
\end{align*}
$$

which is also a Lyapunov coordinate transformation, and define $u_2(t) := \Delta v(t)$. Computing the time derivative of $x_1(t)$ gives the nonlinear system

$$
\begin{align*}
x_1(t) &= u_2(t) \\
y(t) &= \begin{bmatrix} \|x_1(t)\| \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_1(t) \end{bmatrix},
\end{align*}
$$

(3)

where $x_1(t) \in \mathbb{R}^3$ is the system state, $u_2(t) \in \mathbb{R}^3$ is the system input, and $y(t) \in \mathbb{R}^2$ is the system output.

The problems considered in this paper are the observability analysis of the nonlinear systems (2) and (3) and the design of state observers for those systems.

### III. OBSERVABILITY ANALYSIS

This section details the observability analysis of (2) and (3) through state augmentation. With the proposed methodology, it is possible to derive linear systems which capture the behavior of the nonlinear systems, and as such study their observability in a linear systems framework.

#### A. State Augmentation (first case)

To derive a linear system that mimics the dynamics of the nonlinear system (2), define three additional scalar state variables as

$$
\begin{align*}
x_3(t) &:= y_1(t) \\
x_4(t) &:= x_1^T(t)x_2(t) \\
x_5(t) &:= \|x_2(t)\|^2
\end{align*}
$$

and denote

$$x(t) := \begin{bmatrix} x_1^T(t) & x_2^T(t) & x_3(t) & x_4(t) & x_5(t) \end{bmatrix}^T \in \mathbb{R}^n, \quad n = 9,
$$

the augmented state. Then, it can be shown that the dynamics of the augmented system can be written as

$$
\begin{align*}
\dot{x}(t) &= A(t)x(t) + Bu_1(t) \\
y(t) &= Cx(t)
\end{align*}
$$

(4)

where

$$A(t) = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & u_2^T(t) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix},
$$

and $C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The dynamic system (4) can be regarded as a LTV system, even though $A(t)$ depends on the system input and output (see Lemma 1, [4]), and its observability will be analyzed using linear systems theory, applied to the pair $(A(t), C)$.

**Remark 1:** Notice that there is nothing in (4) imposing

$$
\begin{align*}
y_1(t) &= \|x_1(t)\| \\
x_4(t) &= x_1^T(t)x_2(t) \\
x_5(t) &= \|x_2(t)\|^2
\end{align*}
$$

so the observability of (4) does not automatically entail the observability of (2).

### B. Observability of the linear system (first case)

Before showing a sufficient condition for the observability of (4), it is convenient to compute the transition matrix of $A(t)$. To simplify the derivation of results, define

$$u_1^{[2]}(t, t_0) := \int_{t_0}^{t} u_1(\sigma)d\sigma = \begin{bmatrix} u_{11}^{[2]}(t, t_0) \\ u_{12}^{[2]}(t, t_0) \\ u_{13}^{[2]}(t, t_0) \end{bmatrix} \in \mathbb{R}^3.
$$

Then, the transition matrix is given by

$$\phi(t, t_0) = \begin{bmatrix} \phi_1^T(t, t_0) & \phi_2^T(t, t_0) & \phi_3^T(t, t_0) & \phi_4^T(t, t_0) \end{bmatrix}^T,
$$

where

$$\phi_1(t, t_0) = \begin{bmatrix} [I \ (t-t_0)I & 0 & 0 & 0 \end{bmatrix},
$$

$$\phi_2(t, t_0) = \begin{bmatrix} 0 & I & 0 & 0 & 0 \end{bmatrix},
$$

and $\phi_3(t, t_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$

The observability Gramian for $(A(t), C)$ is given by

$$W(t_0, t_f) = \int_{t_0}^{t_f} \phi^T(t, t_0)C^T C\phi(t, t_0)dt.
$$

Note that, due to the specific structure of $C$, the above expression can be reduced to

$$W(t_0, t_f) = \int_{t_0}^{t_f} \psi^T(t, t_0)\psi(t, t_0)dt,
$$

(5)

where

$$\psi(t, t_0) = \begin{bmatrix} \phi_1(t, t_0) \\ \phi_2(t, t_0) \\ \phi_3(t, t_0) \end{bmatrix},
$$

in which $\phi_3(t, t_0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ $\phi_1(t, t_0)$. The following result presents a sufficient condition for the observability of (4).

**Theorem 1:** Suppose that the set of functions

$$\mathcal{F} = \left\{(t-t_0), (t-t_0)^2, u_{11}^[(t-t_0); t_0], u_{12}^[(t-t_0]; t_0), \right. \left. (t-t_0)^3, u_{11}^[(t-t_0); t_0], (t-t_0)^2 u_{12}^[(t-t_0); t_0] \right\}
$$

(6)

is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. Then, the LTV system (4) is observable on $[t_0, t_f]$ in the sense that, given the system input $u_1(t), t \in [t_0, t_f]$, and the system output $y(t), t \in [t_0, t_f]$, the initial condition is uniquely defined.

**Proof:** Suppose that the LTV system (4) is not observable on $[t_0, t_f]$. Then, the observability Gramian $W(t_0, t_f)$ is not positive definite and therefore

$$\exists \forall \quad : d^T W(t_0, t_f) d = 0.
$$

$$d \in \mathbb{R}^n \quad t \in [t_0, t_f] \quad \|d\| = 1.
$$

(7)
Let \( d = [d_{11} \ d_{12} \ d_{13} \ d_{21} \ d_{22} \ d_{23} \ d_3 \ d_4 \ d_5]^T \in \mathbb{R}^n \).

Expanding (7) and using (5) gives
\[
\int_{t_0}^t [\psi(\sigma, t_0) d]^T \psi(\sigma, t_0) d \sigma = 0, \ \forall \ t \in [t_0, t_f],
\]
and it follows that
\[
\begin{align*}
\phi_2(t, t_0) d &= 0 \\
\phi_3(t, t_0) d &= 0,
\end{align*}
\]
for all \( t \in [t_0, t_f] \). But \( \phi_2(t_0, t_0) d = d_3 \), so for (8) to hold, it must be \( d_3 = 0 \). And, since \( \phi_3(t_0, t_0) d = d_{13} \), it follows that \( d_{13} = 0 \). From (8), it also follows that
\[
\frac{d}{dt} \phi_3(t, t_0) d = 0, \ \forall t \in [t_0, t_f],
\]
which means that \( d_{23} = 0 \), and also that
\[
0 = u_{11}(t)d_{11} + u_{12}(t)d_{12} + [u_1(t, t_0) + (t-t_0)u_{11}(t)]d_{21} + [u_2(t, t_0) + (t-t_0)u_{12}(t)]d_{22} + d_4 + (t-t_0)d_5
\]
for all \( t \in [t_0, t_f] \). Integrating both sides of (9) gives
\[
0 = u_{11}(t, t_0)d_{11} + u_{12}(t, t_0)d_{12} + (t-t_0)u_{11}(t, t_0)d_{21} + (t-t_0)u_{12}(t, t_0)d_{22} + (t-t_0)d_4 + \frac{(t-t_0)^2}{2}d_5,
\]
which implies that the set of functions \( \mathcal{F} \) is not linearly independent on \([t_0, t_f]\) if it is linearly dependent on \([t_0, t_f]\). The observability Gramian must be positive definite, and therefore (4) is observable (see Lemma 1, [4]).

C. Observability of the Nonlinear System (first case)

The following theorem provides a sufficient condition for the observability of the nonlinear system (2), as well as a practical result that can be used in the design of state observers for that system.

**Theorem 2**: Suppose that the set of functions (6) is linearly independent on \([t_0, t_f]\), \( t_0 < t_f \). Then, the nonlinear system (2) is observable on \([t_0, t_f]\) in the sense that, given the system input \( u_1(t), \ t \in [t_0, t_f] \), and the system output \( y(t), \ t \in [t_0, t_f] \), the initial condition is uniquely defined. Moreover, a state observer with globally asymptotically stable error dynamics for the LTV system (4) is also a state observer for the nonlinear system (2), with globally asymptotically stable error dynamics.

**Proof**: Let
\[
\begin{bmatrix}
x_1(t_0) \\
x_2(t_0)
\end{bmatrix} = \begin{bmatrix} [x_{11}(t_0) \ x_{12}(t_0) \ x_{13}(t_0)]^T \\
x_{21}(t_0) \ x_{22}(t_0) \ x_{23}(t_0)
\end{bmatrix}
\]
be the initial state of the nonlinear system (2). Then,
\[
y_2(t) = x_{13}(t) + (t-t_0)x_{23}(t_0) + u_1(t_0).
\]
Taking the square of \( y_1(t) \), it follows that
\[
y_1^2(t) = ||x_1(t_0)||^2 + ||x_2(t_0)||^2(t-t_0)^2 + \frac{u_1(t_0, t_0)}{2} + 2(t-t_0)x_1^T(t_0)x_2(t_0) + 2x_1^T(t_0)u_1(t_0) + 2(t-t_0)x_1^T(t_0)u_1(t_0).
\]

Since the set of functions (6) is assumed linearly independent on \([t_0, t_f]\) it follows, from Theorem 1, that the LTV system (4) is observable on \([t_0, t_f]\). Thus, given \( y(t) \) and \( u_1(t) \) for \( t \in [t_0, t_f] \), the initial state of (4) is determined uniquely. Let
\[
\begin{bmatrix}
x_1(t_0) \\
x_2(t_0) \\
x_3(t_0) \\
x_4(t_0) \\
x_5(t_0)
\end{bmatrix} = \begin{bmatrix} [x_{11}(t_0) \ x_{12}(t_0) \ x_{13}(t_0)]^T \\
x_{21}(t_0) \ x_{22}(t_0) \ x_{23}(t_0)
\end{bmatrix}
\]
be the initial state of the linear system (4). Then,
\[
y_2(t) = x_{13}(t_0) + (t-t_0)x_{23}(t_0) + u_1(t_0).
\]
As \( y_1^2(t) = \frac{2t^2}{3} \), it follows that
\[
\frac{d}{dt} y_1^2(t) = 2t x_1(t_0) + 2t x_2(t_0) + x_1(t_0) + x_2(t_0)
\]
and from that
\[
y_1^2(t) = 2x_1^T(t_0)u_1(t_0) + 2(t-t_0)x_2^T(t_0)u_1(t_0) + 2(t-t_0)x_1(t_0)u_1(t_0) + 2x_1(x_2(t_0) - x_1(t_0))u_1(t_0) + 2(t-t_0)x_2(t_0)u_1(t_0) + 2(t-t_0)x_1(t_0)u_1(t_0)
\]
for all \( t \in [t_0, t_f] \). Since the set of functions \( \mathcal{F} \) is assumed linearly independent, (14) implies that
\[
\begin{align*}
x_1(t_0) &= x_1(t_0) \\
x_2(t_0) &= x_2(t_0) \\
x_1^T(t_0)x_2(t_0) &= 0 \\
||x_2(t_0)||^2 &= x_5(t_0)
\end{align*}
\]
This concludes the proof, as both the initial state of the nonlinear system (2) is uniquely determined and the initial state of the linear system (4) matches the initial state of the nonlinear system.

D. Observability analysis (second case)

This subsection establishes results for the nonlinear system (3) similar to those shown in previous subsections. In this case, the system is simpler, and therefore only one additional scalar state variable is needed, \( x_2(t) := y_1(t) \), and the augmented state is \( x(t) := [x_1(t) \ x_2(t)] \in \mathbb{R}^{n_2}, n_2 = 4 \). The dynamics of the augmented system can be written in the form
\[
\begin{align*}
\dot{x}(t) &= A_2(t)x(t) + B_2u_2(t) \\
y(t) &= C_2x(t)
\end{align*}
\]
where \( A_2(t) = \begin{bmatrix} 0 & 0 \\ \mathbf{u}_2(t) & 0 \\ v_1(t) & 0 \end{bmatrix} \), \( B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), and \( C_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 \end{bmatrix} \). As in the previous case, this system can be regarded as a LTV system, and its observability will be analyzed in a similar way. Note that the nonlinear system (3) is a special case of (2), with \( x_2(t_0) = 0 \), so the results of the previous sections are valid for (3). Nevertheless, a less restrictive sufficient condition can be found, as it will be detailed in the rest of this subsection. Before providing a sufficient condition for the observability of (15), it is convenient to define

\[
\mathbf{u}_2^{[1]}(t, t_0) := \int_{t_0}^{t} \mathbf{u}_2(\sigma)d\sigma = \begin{bmatrix} u_2^{[1]}(t, t_0) \\ u_2^{[1]}(t, t_0) \\ u_2^{[1]}(t, t_0) \end{bmatrix} \in \mathbb{R}^3.
\]

The following theorem provides a sufficient condition for the observability of (15).

**Theorem 3:** Suppose that the set of functions

\[
\mathcal{F}_2 = \{ u_2^{[1]}(t, t_0), u_2^{[2]}(t, t_0) \}
\]

is linearly independent on \([t_0, t_f], t_0 < t_f\). Then, the LTV system (15) is observable on \([t_0, t_f] \) in the sense that, given the system input \( u_2(t), t \in [t_0, t_f] \), and the system output \( y(t), t \in [t_0, t_f] \), the initial condition is uniquely defined.

The following theorem provides a sufficient condition for the observability of the nonlinear system (3), as well as a practical result that can be used in the design of state observers for that system.

**Theorem 4:** Suppose that the set of functions (16) is linearly independent on \([t_0, t_f], t_0 < t_f\). Then, the nonlinear system (3) is observable on \([t_0, t_f] \) in the sense that, given the system input \( u_2(t), t \in [t_0, t_f] \), and the system output \( y(t), t \in [t_0, t_f] \), the initial condition is uniquely defined. Moreover, a state observer with globally asymptotically stable error dynamics for the LTV system (15) is also a state observer for the nonlinear system (3), with globally asymptotically stable error dynamics.

**IV. SIMULATION RESULTS**

This section provides simulation results to assess the performance of the filtering solutions proposed in the paper. Due to space constraints, only the first case, in which the AUV has access to the ASC’s inertial velocity, is considered.

While the observability analysis was carried out in the inertial coordinate frame, it is possible to design a GAS state estimator in the original coordinate space by applying the appropriate state transformation. Let

\[
\mathbf{T}(t) = \text{diag} \left( \mathbf{R}^T(t), -\mathbf{R}^T(t) \right) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 3},
\]

and define new state variables \( \mathbf{z}(t) \in \mathbb{R}^n \) such that \( \mathbf{z}(t) = \mathbf{T}(t)\mathbf{x}(t) \). Then, it is straightforward to show that the dynamics of the new state \( \mathbf{z}(t) \) can be described by the LTV system

\[
\begin{aligned}
\dot{\mathbf{z}}(t) &= \mathbf{A}_b(t)\mathbf{z}(t) + \mathbf{B}_b(t)\mathbf{u}_1(t) \\
\mathbf{y}(t) &= \mathbf{C}_b(t)\mathbf{z}(t),
\end{aligned}
\]

where \( \mathbf{A}_b(t) = [\mathbf{T}(t) + \mathbf{T}(t)\mathbf{a}(t)]\mathbf{T}^T(t), \mathbf{B}_b(t) = \mathbf{T}(t)\mathbf{B}, \) and \( \mathbf{C}_b(t) = \mathbf{C}\mathbf{T}^T(t) \). Notice that, as \( \mathbf{T}(t) \) is a Lyapunov state transformation, the stability and observability properties of (4) also apply to (17) [5]. Following this, one can use Kalman filtering to design a GAS state observer for (17) [8]. As for the parameters of the simulation, the AUV starts at \( s(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \) (m), and its inertial velocity follows

\[
\mathbf{v}_s(t) = \begin{bmatrix} 1.5 \sin(2\pi ft) \\ 0.5 \cos(2\pi ft) \\ 0 \end{bmatrix} \text{ (m/s)}, \ f = 0.005 \text{ (s)}^{-1}.
\]

The AUV starts at \( \mathbf{p}(0) = \begin{bmatrix} 0 & 25 \end{bmatrix}^T \) (m), and is assumed to be moving in a fluid with velocity \( \mathbf{v}_f = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}^T \) (m/s) relative to \( \{I\} \), expressed in inertial coordinates. The velocity of the I-AUV relative to the fluid, also in inertial coordinates, follows \( \mathbf{v}_r(t) = \begin{bmatrix} 1.5 & -0.5 \end{bmatrix}^T \) (m/s). The trajectories described by the ASC and the I-AUV are depicted in Fig. 2.

**Remark 2:** The peculiar trajectory of the ASC was chosen to guarantee observability. In fact, in many practical settings the AUV will be required to follow a certain desired path, and as such the trajectory of the ASC should be chosen such as to guarantee observability, preferably leaving a wide margin for possible perturbations.

The measurement noise was simulated by adding a zero-mean, uncorrelated and normally distributed perturbation to the velocity, range, and depth measurements, with standard deviations of 0.5 m/s, 0.01 m, and 0.01 m, respectively.
with practical interest, $x_1$ and $x_2$. As it can be seen, even in the presence of sensor noise with realistic intensity, the achieved values remain confined to small intervals around zero and excellent filtering performance is achieved.

V. CONCLUSIONS

This paper proposed novel cooperative navigation solutions for an I-AUV working in tandem with an ASC. The I-AUV was assumed to be moving in the presence of constant unknown ocean currents, and to have access to measurements of its range to the ASC and of its depth relatively to the sea level to estimate its position. Two different scenarios were considered: in one, the ASC transmits its position and velocity to the I-AUV, while in the other the ASC transmits only its position, and the I-AUV has access to measurements of its velocity relative to the ASC. A sufficient condition for observability and a method for designing state observers with GAS error dynamics were presented for both cases. Finally, simulation results were presented and discussed to assess the performance of the proposed solutions in the presence of measurement noise.

REFERENCES


![Graphical data](image)

**Fig. 4.** Detailed view of the steady-state estimation error variables

The results of the simulation are depicted in Fig. 3, Fig. 4, and Table I. Fig. 3 details the evolution of the estimation error variables. The large transients that can be observed are caused by mismatches in the initial conditions. To better assess the performance of the proposed state estimator, Fig. 4 depicts the detailed evolution of the estimation error variables after the initial transients have settled. To complement the graphical data, Table I details the measured standard deviations of the steady-state estimation error of the variables.

**TABLE I**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1(t)$</td>
<td>$3.05 \times 10^{-2}$ (m)</td>
</tr>
<tr>
<td>$z_2(t)$</td>
<td>$4.12 \times 10^{-2}$ (m)</td>
</tr>
<tr>
<td>$z_3(t)$</td>
<td>$0.91 \times 10^{-2}$ (m)</td>
</tr>
<tr>
<td>$z_4(t)$</td>
<td>$3.95 \times 10^{-4}$ (m/s)</td>
</tr>
<tr>
<td>$z_5(t)$</td>
<td>$2.55 \times 10^{-4}$ (m/s)</td>
</tr>
<tr>
<td>$z_6(t)$</td>
<td>$3.23 \times 10^{-4}$ (m/s)</td>
</tr>
</tbody>
</table>

![Table I](image)